Fringe Effects in MAD
PART I
Second Order Fringe in MAD-X for the Module PTC

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Abstract

The various versions of MAD contain a second order fringe field effect for the ideal bend. The expressions appeared in the famous SLAC-75 report where it is reported that they were derived by Hindmarsh and Brown in an unpublished work. We would like to rederive these expressions and check them. Moreover, we want to create maps for them which are not expansions around a "design orbit" but true operators valid around arbitrary orbits or at least a larger class of orbits. As in the original SLAC-75[1] report, our expression are expansions and are not meant to replace precise integration. We simply want a knob which is "PTC-compliant," that is to say, exact in the Talman sense. Such PTC compliant maps have the virtue that they could be replaced by a thin exact "sandwich" made out of exact integrators. In fact the numerical checks of the SLAC-75 results, we precisely made out of such integrators.

Contents

A Discussion of the SLAC-75 Expressions

A.1 The SLAC-75 Expressions

In SLAC-75, the linear vertical focusing at the entrance of a bend is given by

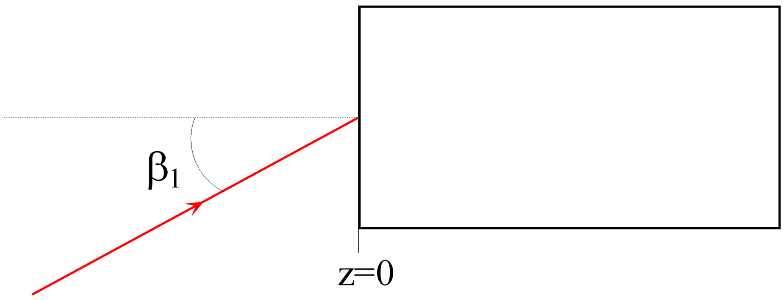

Figure 1: Rectangular Bend and Design Trajectory Geometry

$$p_y^f = p_y^i + \left\{ -b_0 \tan(\beta_1) + \frac{b_0^2 g K}{\cos^3(\beta_1)} \left(1 + \sin^2(\beta_1)\right) \right\} y \tag{1}$$

where K is defined as

$$K = \int_{-\infty}^{+\infty} \frac{b(z)\,(b_0 - b(z))}{g b_0^2}\, dz \tag{2}$$

Here g is the height of the vertical gap in the dipole. The quantity $b(z)$ is the field scaled by the design p_0/q, i.e., the so-called $B\rho$. Here is a typical plot of $b(z)/b_0$:

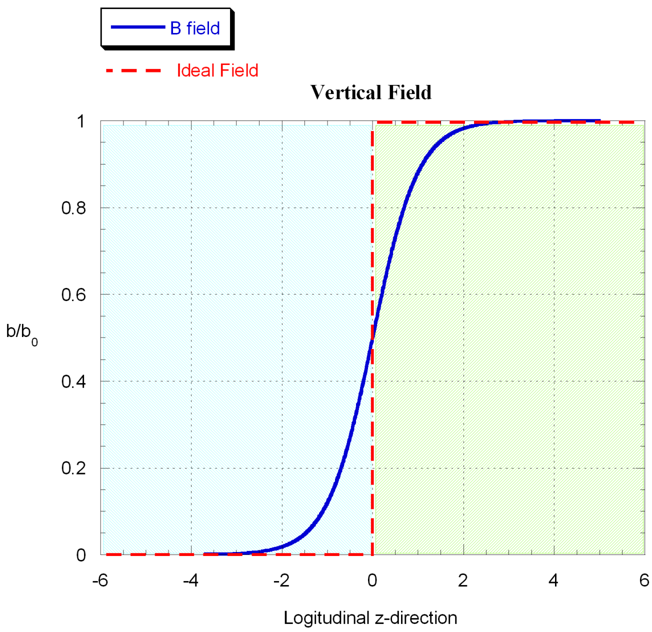

Figure 2: Example of a Field Profile for B_y

A.2 Real Calculation of a Fringe Effect

In our work we integrated some real field and compared it to something like Equation (1). Using the "Fully Polymorphic Package" one can easily generate a Taylor series matrix for the edge by sandwiching the exact map between an inverse drift and an inverse ideal bend. Therefore, calling T, the map of an edge, we must have:

$$T = B_{\varepsilon\to 0}(b_0) \circ F_{-\varepsilon\to\varepsilon} \circ D_{-\varepsilon} \tag{3}$$

Here B is the transfer map for a constant b_0 and it is generated by the Hamiltonian:

$$H = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + b_0 x \tag{4}$$

The drift is the above Hamiltonian with $b_0 = 0$. The Hamiltonian in the fringe region is given by:

$$\begin{aligned} H &= -\sqrt{(1+\delta)^2 - (p_x - a)^2 - p_y^2} + b(z)\ x \\ a &= \sum_{n=1}^{\infty} \frac{(-1)^n b^{[2n-1]}}{(2n)!} y^{2n} \quad \text{where } b^{[2n-1]} = \frac{d^{2n-1}b}{dz^{2n-1}}. \end{aligned} \tag{5}$$

We checked our work and the formulas of SLAC-75 using the following model:

$$\begin{aligned} b(z) &= b_0 \frac{1 + tanh(z/\delta)}{2} \\ b'(z) &= b_0 \frac{1 - tanh^2(z/\delta)}{2\delta} \\ b''(z) &= b_0 \frac{-tanh(z/\delta)\left(1 - tanh^2(z/\delta)\right)}{\delta^2} \\ Kg &= \frac{\delta}{2} \end{aligned} \tag{6}$$

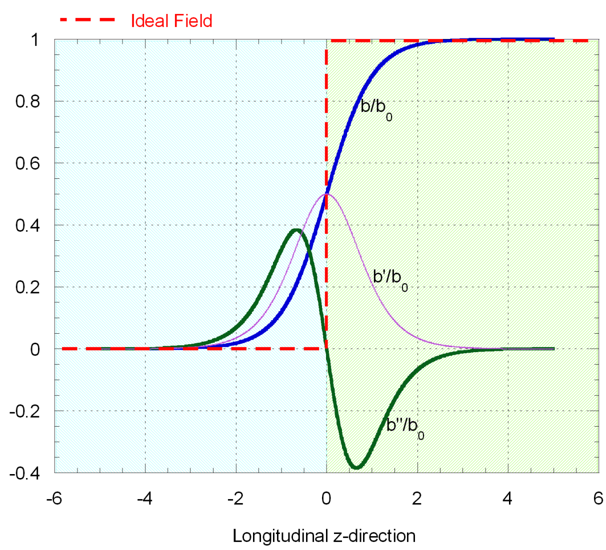

Figure 3: B_y and few relevant derivatives

The exact integration in Equation (3) and the SLAC-75 result do agree if we substitute for the integral Kg the value $\delta/2$.

N.B. Presumably, the result of SLAC-75 include a pole face rotation of angle β_1 that rotates the pole face coordinates into the coordinates of the design trajectory. This transformation, the so-called PROT of Dragt or the ROT_XZ of the code PTC, does not affect the vertical plane in leading order. This is why it is not mentioned in our discussion.

A.3 Calling it "Quits"

The expression of Equation (1) is not PTC-compliant because it is obvious that β_1 is not a property of the magnet but it just related to the incoming momentum, that is to say, it is the angle of a trajectory which happens to be the so-called design trajectory in a code like MAD. In PTC we would like the expression to work for any trajectory. Thus given our confidence in SLAC-75's expressions, as confirmed by the integration described in Sect. A.2, we can create the following generating function for the thin fringe:

$$\begin{aligned} F &= p_x x^f + p_y y^f + \Delta \ell^f - \frac{1}{2}\left\{-b_0 \frac{p_x}{p_z} + \frac{b_0^2 g K}{p_z^3}\left(1+p_x^2\right)\right\} {y^f}^2 \\ \text{where } p_z &= \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \text{ and } \Delta = -\delta. \end{aligned} \tag{7}$$

This mixed generating function will reproduce the result of Equation (1) as well as introduced nonlinear changes not inconsistent with the correct dynamical structure of this system.

It turns out as well will see, that Equation (7) contains most of the physics one can hope to extract from a simple fringe second order calculation. We now prove this, for the record, in the next section.

B Operator Derivation of SLAC-75 Fringe Effects

In this section we derive the analytical expression related to SLAC-75. We do this for the entrance fringe region because the calculation is cleaner. Indeed if we concentrate on the action of the total map on the variables p_y, then for the factorization of Equation (16), the following is true:

$$\begin{aligned} \mathcal{T} p_y &= \mathcal{S}\mathcal{D}_\varepsilon \mathcal{B}_{\varepsilon\to 0}(b_0) p_y \\ &= \mathcal{S} p_y \end{aligned} \tag{8}$$

Therefore, the terms coming from the ideal (no fringe) magnet do not contribute. The exit map can be obtained from the formula for the entrance map by looking at the inverse of the entrance map. The formula is then the same except that the sign of b_0 is reversed. This seems to contradict SLAC-75, however one must also remember that the sign of the angle at the exit face is defined differently: the tangent of the exit angle is really the negative of the variable x' at the exit face.

B.1 Preliminary Manipulations

Following Dragt and others, we write an operator equation for the map F. Actually, if we denote the Lie map with the curly letter $\mathcal{F}$, it must obeys the equation

$$\frac{d\mathcal{F}}{dz} = \mathcal{F} : -H : \tag{9}$$

We can use the Heisenberg representation and write $\mathcal{F}$ as follows:

$$\mathcal{F} = \mathcal{P}\mathcal{D} \tag{10}$$

Here $\mathcal{D}$ is the drift map and $\mathcal{P}$ is the residual effect of the bend. Obviously $\mathcal{D}$ obeys the equation:

$$\frac{d\mathcal{D}}{dz} = \mathcal{D} : -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} : \tag{11}$$

As for $\mathcal{P}$, it obeys the usual interaction picture equation as in quantum mechanics:

$$\begin{aligned} \frac{d\mathcal{P}}{dz} &= \mathcal{P}\mathcal{D} : -V : \mathcal{D}^{-1} \\ V &= H + \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \\ &= \sum_{n=1}^{\infty} b_0^n V_n \ . \end{aligned} \tag{12}$$

Since, according to Equation (3), the map is to be integrated from $z = -\varepsilon$, the map $\mathcal{D}$ in Equation (12) is a drift from $z = -\varepsilon$ to an arbitrary position z:

$$\begin{aligned} \mathcal{D}x &= x + (z+\varepsilon)x' \\ \mathcal{D}y &= y + (z+\varepsilon)y' \end{aligned} \tag{13}$$

In Equation (12) all maps and operators are symplectic, and therefore the following is true:

$$\begin{aligned} \frac{d\mathcal{P}}{dz} &= \mathcal{P} : -V^\dagger : \\ \text{where } V^\dagger(x,y;z) &= V(x + (z+\varepsilon)x', y + (z+\varepsilon)y'; z) \end{aligned} \tag{14}$$

To reproduce SLAC-75's results, we need to solve Equation (14) to second order. This can be done by integrating both sides of the equation from $z = -\varepsilon$ to z:

$$\begin{aligned} \int_{-\varepsilon}^{z} \frac{d\mathcal{P}}{dz} dz' &= \int_{-\varepsilon}^{z} \mathcal{P} : -V^\dagger_{z'} : dz' \\ \Rightarrow \mathcal{P} &= 1 + \int_{-\varepsilon}^{z} \mathcal{P} : -V^\dagger_{z'} : dz' \\ \Longrightarrow \mathcal{P}_{-\varepsilon\to\varepsilon} &= 1 + \int_{-\varepsilon}^{\varepsilon} : -V^\dagger_{z'} : dz' + \int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' : V^\dagger_{z''} :: V^\dagger_{z'} : \end{aligned} \tag{15}$$

We can rewrite Equation (3) for Lie operators as

$$\begin{aligned} \mathcal{T} &= \mathcal{D}_{-\varepsilon}\mathcal{F}_{-\varepsilon\to\varepsilon}\mathcal{B}_{\varepsilon\to 0}(b_0) \\ &= \mathcal{D}_{-\varepsilon}\mathcal{P}\mathcal{D}_{2\varepsilon}\mathcal{B}_{\varepsilon\to 0}(b_0) \\ &= \mathcal{S}\mathcal{D}_{\varepsilon}\mathcal{B}_{\varepsilon\to 0}(b_0) \end{aligned} \tag{16}$$

$\mathcal{S}$ is given by

$$\begin{aligned} \mathcal{D}_{-\varepsilon}\mathcal{P}_{-\varepsilon\to\varepsilon}\mathcal{D}_{\varepsilon} &= 1 + \overbrace{\int_{-\varepsilon}^{\varepsilon} : -W_{z'} : dz'}^{\text{linear term in } V} + \overbrace{\int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' : W_{z''} :: W_{z'} :}^{\text{second order term in } V} \\ \text{where} \quad W_z &= V(x + zx', y + zy'; z) \end{aligned} \tag{17}$$

B.2 Actual Calculation

Equation (17) is really the time ordered exponential found in standard quantum mechanics applied to the Heisenberg representation. Anyone familiar with these techniques can write it down immediately. So the real calculation starts here.

We decide to do the calculation no higher than second order in b_0 and at most to y^4. Having this in mind, we can now write down the function W:

$$\begin{aligned} W_z &= \overbrace{b\widetilde{x} - x' \underbrace{\left\{ -\frac{b'}{2}\widetilde{y}^2 + \frac{b'''}{4!}\widetilde{y}^4 + \cdots \right\}}_{a}}^{\text{First order in } b} - \frac{b'^2}{8p_z}\left\{ 1 + x'^2 \right\}\widetilde{y}^4 \\ \widetilde{y} &= y + zy' \end{aligned} \tag{18}$$

The first term to consider is the linear term in Equation (17). The first term is not important as it mostly changes the orbit:

$$\int_{-\varepsilon}^{\varepsilon} b\widetilde{x} dz = x \int_{-\varepsilon}^{\varepsilon} b\, dz \; + \; x' \int_{-\varepsilon}^{\varepsilon} bz\, dz \tag{19}$$

Then the term proportional to a can be integrated exactly to all orders!

$$\int_{-\varepsilon}^{\varepsilon} x' a_z dz \quad = \quad -\frac{b_0}{2} \frac{x'}{1+y'^2} y^2 \tag{20}$$

This result is obtained by integrating by parts every term in the expansion for a until one gets down to b'. It is just the standard result expect for the funny dependence on y'. The next term is

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{8p_z} \left\{ 1+x'^2 \right\} b'^2 \widetilde{y}^4 dz \tag{21}$$

Expanding $\widetilde{y}$ $(= y + zy')$ and integrating, we see that Equation (21) requires only the following integrals:

$$I_n \quad = \quad \int_{-\varepsilon}^{\varepsilon} b'^2 z^n \; dz \qquad n = 0, \cdots, 4 \tag{22}$$

The terms from Equation (21) are all nonlinear, i.e., $\propto y^4$,

for particles in the mid-plane. We will not study them any further.

The second term in Equation (17) is far more complex! We will examine the effect on the momentum p_y. To second order in b_0, we have:

$$\Delta p_y \quad = \quad \int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' : x' a_{z''} :: x' a_{z'} : p_y \tag{23}$$

We now concentrate temporarily on the leading term of this expansion:

$$\begin{aligned}
\Delta p_y^{11} \quad &= \quad \int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' : b_{z''} \left(x + z'' x' \right) + x' \frac{b'_{z''}}{2} \left(y + z'' y' \right)^2 :: b_{z'} \left(x + z' x' \right) + x' \frac{b'_{z'}}{2} (y + z' y')^2 : p_y \\
&= \quad \int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' : \overbrace{b_{z''} \left(x + z'' x' \right)}^{A} + \overbrace{x' \frac{b'_{z''}}{2} \left(y + z'' y' \right)^2}^{B} : \overbrace{\left\{ x' b'_{z'} \left(y + z' y' \right) \right\}}^{C} \\
&= \quad \int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' \left\{ [A, C] + [B, C] \right\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
[A, C] \quad &= \quad b'_{z'} b_{z''} \left\{ [x, x'] \left(y + z' y' \right) + x' \left(z' - z'' \right) [x, y'] \right\} \\
[B, C] \quad &= \quad \frac{b'_{z'} b'_{z''}}{2} \left\{ x' \, [y, x'] \left\{ y^2 + \left(2z' z'' - z''^2 \right) y'^2 + 2z' y y' \right\} \right. \\
&\qquad \left. + x'^2 \, [y, y'] \left\{ 2y \left(z' - z'' \right) + 2y' \left(z' - z'' \right) y'^2 \right\} \right\}
\end{aligned} \tag{25}$$

In an attempt to get the SLAC-75 results, we retain only the terms proportional to y in Equation (25).

$$d\Delta p_y^{11}/dy\big|_{y=0} \quad = \quad \int_{-\varepsilon}^{\varepsilon} dz' \int_{-\varepsilon}^{z'} dz'' \left\{ b'_{z'} b_{z''} \, [x, x'] + x'^2 \, [y, y'] \, b'_{z'} b'_{z''} \left(z' - z'' \right) \right\} \tag{26}$$

According to Equation (26), we must perform 2 integrals:

$$\begin{aligned}
J_0 \quad &= \quad \int_{-\varepsilon}^{\varepsilon} b'_{z'} dz' \int_{-\varepsilon}^{z'} b_{z''} dz'' \text{ and } J_{10} - J_{01} \\
\text{where } J_{10} \quad &= \quad \int_{-\varepsilon}^{\varepsilon} b'_{z'} z' dz' \int_{-\varepsilon}^{z'} b'_{z''} dz'' \\
\text{and } J_{01} \quad &= \quad \int_{-\varepsilon}^{\varepsilon} b'_{z'} dz' \int_{-\varepsilon}^{z'} b'_{z''} z'' dz''
\end{aligned} \tag{27}$$

Let us look at J_0 first:

$$
\begin{aligned}
J_0 &= \int_{-\varepsilon}^{\varepsilon} b'_{z'}dz' \int_{-\varepsilon}^{z'} b_{z''}dz'' \\
&= b_{z'} \int_{-\varepsilon}^{z'} b_{z''}dz'' \Big|_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} b_{z'}^2 dz' \\
&= \int_{-\varepsilon}^{\varepsilon} b_z\,(b_0 - b_z)\,dz \; = g b_0^2 K
\end{aligned}
\tag{28}
$$

In Equation (28), we assume that the asymptotic values have been reached, i.e., $b(-\varepsilon) = 0$ and $b(\varepsilon) = b_0$. We now proceed with J_{10}:

$$
\begin{aligned}
J_{10} &= \int_{-\varepsilon}^{\varepsilon} b'_{z'}z'dz' \int_{-\varepsilon}^{z'} b'_{z''}dz'' \;=\; \int_{-\varepsilon}^{\varepsilon} b'_z b_z z\; dz \;=\; \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} \frac{db_z^2}{dz}\; z\; dz \\
&= \frac{1}{2} b_z^2 z\Big|_{-\varepsilon}^{\varepsilon} - \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} b_z^2\; dz \;=\; \frac{1}{2} b_0^2 \varepsilon \; - \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} b_z^2\; dz
\end{aligned}
\tag{29}
$$

and J_{01}:

$$
\begin{aligned}
J_{01} &= \int_{-\varepsilon}^{\varepsilon} b'_{z'}dz' \int_{-\varepsilon}^{z'} b'_{z''}z''dz'' \;=\; b_{z'} \int_{-\varepsilon}^{z'} b'_{z''}z''dz''\Big|_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} b_z b'_z\; zdz \\
&= b_0 \int_{-\varepsilon}^{\varepsilon} b'_z z\; dz - \int_{-\varepsilon}^{\varepsilon} b_z b'_z\; z \;=\; b_0 \int_{-\varepsilon}^{\varepsilon} b'_z z\; dz - J_{10} \\
&= b_0 b_z z|_{-\varepsilon}^{\varepsilon} - b_0 \int_{-\varepsilon}^{\varepsilon} b_z\; dz - J_{10} \;=\; b_0^2 \varepsilon \;-\; b_0 \int_{-\varepsilon}^{\varepsilon} b_z\; dz - J_{10}
\end{aligned}
\tag{30}
$$

Thus we have

$$
J_{10} - J_{01} \;=\; b_0 \int_{-\varepsilon}^{\varepsilon} b_z\; dz - \int_{-\varepsilon}^{\varepsilon} b_z^2\; dz \;=\; g b_0^2 K
\tag{31}
$$

We are now ready to compute the total second order effect given by Equation (26):

$$
\begin{aligned}
d\Delta p_y^{11}/dy\big|_{y=0} &= g b_0^2 K \left\{ [x,x'] + {x'}^2\, [y,y'] \right\} \\
&= g b_0^2 K \left\{ \underbrace{\frac{(1+\delta)^2 - p_y^2}{p_z^3}}_{[x,x']} + \underbrace{\frac{p_x^2}{p_z^2}}_{x'^2} \underbrace{\frac{(1+\delta)^2 - p_x^2}{p_z^3}}_{[y,y']} \right\}
\end{aligned}
\tag{32}
$$

This results should be compared with the "phenomenological" result of Sect. A.3, Equation (7). Indeed if we substitute $\delta = 0$, $p_y = 0$, and $p_x = \sin(\beta_1)$, then it agrees perfectly with the SLAC-75 result at least as far as linear terms are concerned.

C PTC Implementation of SLAC-75 Fringe Effects

C.1 Our initial PTC implementation

In the initial PTC implementation , we constructed the following generating function (as in Equation (7)):

$$
F \;=\; p_x x^f + p_y y^f + \Delta\ell^f - \frac{1}{2}\Phi(p_x, p_y, \delta) {y^f}^2
\tag{33}
$$

$$
\text{where } \Phi(p_x, p_y, \delta) \;=\; \frac{b_0 x'}{1 + {y'}^2} \;-\; g b_0^2 K \left\{ \underbrace{\frac{(1+\delta)^2 - p_y^2}{p_z^3}}_{[x,x']} + \underbrace{\frac{p_x^2}{p_z^2}}_{x'^2} \underbrace{\frac{(1+\delta)^2 - p_x^2}{p_z^3}}_{[y,y']} \right\} \text{ and } \Delta = -\delta
\tag{34}
$$

PTC solves this equation at the entrance of any dipole element and a similar one at the exit. The leading order term is compulsory and the second order term proportional to K is optional. The formulas of this paper are only available for dipole in the exact option. The elements using the expanded Hamiltonian (as in TRACYII or SixTrack) do not use any of the formulas of this paper: they use the infamous quadrupole thin lens which *de facto* incorporates the standard linear term computed in this paper. The additional term is added in the same "inexact" way as a function of the design entrance angle which we know now to be physically incorrect.

The results are

$$
\begin{aligned}
y^f &= \frac{2y}{1+\sqrt{1-2\frac{\partial\Phi}{\partial p_y}y}} \\
x^f &= x + \frac{1}{2}\frac{\partial\Phi}{\partial p_x}{y^f}^2 \\
p_y^f &= p_y - \Phi y^f \\
\ell^f &= \ell - \frac{1}{2}\frac{\partial\Phi}{\partial\delta}{y^f}^2
\end{aligned}
\tag{35}
$$

The two remaining variables p_x and δ stay constant.

C.2 Present MAD8 "compliant" implementation

It is quite clear that our derivation and presumably the original SLAC-75 one as well is only valid to second order in b_0. Thus for finite b_0 we expect differences between MAD8 and PTC even around the so-called design orbit. However this can be avoided by using the following generating function:

$$
F = p_x x^f + p_y y^f + \Delta\ell^f - \frac{1}{2}\psi(p_x,p_y,\delta){y^f}^2 \tag{36}
$$

$$
\text{where } \psi(p_x,p_y,\delta) = b_0\tan\left[\tan^{-1}\left(\frac{x'}{1+{y'}^2}\right) - gb_0K\left\{1+x'^2\left(2+y'^2\right)\right\}p_z\right] \text{ and } \Delta=-\delta \tag{37}
$$

One notices that Equations (33) and (36) agree to order b_0^2.

Acknowledgments

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References

[1] K. L. Brown, Technical Report No. 75, SLAC.