Nonlinear Stochastic Moment Maps:

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(Dated: June 10, 2005)

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I. INTRODUCTION

Electron machines, in the simplest situation, can be viewed as mildly nonsymplectic systems under the influence of a stochastic force. This stochastic force can be decomposed into the average (or classical) radiation and its fluctuation. The classical radiation, when added to the normal single particle magnetic forces, creates a deterministic albeit mildly nonsymplectic system. Close to the origin of phase space, away from resonances, the motion is that of a sink, i.e., all the particle go towards the origin on spiral curves whose shape is mostly determined by the original symplectic system. Once the fluctuation are added, they prevent the beam from collapsing into the origin. Instead a statistical equilibrium is reached: the stochastic fluctuations being on average balanced by the deterministic damping.

In this paper we address the issue of computing this distribution in a systematic way when nonlinearities are added. The standard method, which we will call the Chao-Sands[1] method, consists in exploiting the smallness of the damping decrements and postulating that the equilibrium distribution will rest on the invariant of the original symplectic maps.

The Chao method requires that the map be linear and that the damping decrements be small compared to all the linear resonances. It is an approximate linear theory. It is however a very accurate linear theory because in practice, one must be very close to the linear resonance to see any degradation of the computed beam sizes.

Despite its successes, in the context of tracking codes equipped with truncated power series algebra (TPSA), one is better to start with an exact linear theory. In our case, we have developped tools which overloads the "DA" package of Berz and created a polymorphic type in FORTRAN90; this package also contains numerous analysis tools which overload the old LIELIB library of Forest. This implies that the inclusion of TPSA in a tracking code and its analysis are now painless. Given these tools, we are no longer bound to formalisms which are derivable on paper: we can actually implement formalism that are transparent and closer to the real physics. Such formalism are often easier to implement on the computer than approximate formalisms.

Thus one can actually implement, on the computer, the so-called beam envelope formalism. Here, the only approximation is the linearity of the system. This formalism was first implemented in the KEK code SAD by Oide, Ohmi and Hirata. Forest thanks to TPSA/DA later implemented it in his own tracking tools. Most recently it was put into the package PTC and it will be included in MAD-X or any other code linked to PTC such as BMAD.

Given these powerful tools, one is tempted to ask the following questions:

1. Can we extend in practice the beam envelope approach to a full nonlinear moment map or Stratonovitch expansion, i.e., can we actually compute something on a real system or just babble?

2. Near the origin, does the nonlinear generalization of the beam envelope theory makes sense, does it converge to some sensible result?

3. When it does not converge, although the brute force stochastic tracking shows no resonances, can we fix it?

In this paper we will answer these questions. Question 3 will be partly answered. In fact we will see that, in a paradoxical and ironical way, it is often possible to rescue the nonlinear moment approach under the Chao-Sands condition. In other words, the moment map was introduced initially to generalize the Chao-Sands methods. However, in the nonlinear case when the general moment method diverges, the Chao-Sands method points to a palliative.

In the rest of the paper we will illustrate our claims on simple one-degree of freedom maps on which the reader can perform tracking and check our contentions. In fact, even in the absence of DA/TPSA, on such simple system on can use Mathematica, Mapple or any such manipulator to actually perform our calculations. In the end, if these things are to be really useful, they must be linked to a real tracking code and thus to a TPSA style package as we have done in the library PTC.

It is important to realize that this paper could have been written 15 or 20 years ago. However, in those days, it would have been mostly unpractical theoritical babbling since we would not have been able to perform realistic calculations. The example of the paper are simple so as to guide the reader and illustrate the approach. The reader must convince himself that these things are now, thanks to modern computer tools, within the reach of a standard realistic tracking code.

II. THE LINEAR THEORIES: BEAM ENVELOPE AND CHAO'S INTEGRALS

To understand the differences between these two methods and to see clearly how one (Chao) is an aprroximation of the other, let us a look at a simple one-dimensional example. This example could represent the approximate motion in the longitudinal plane. First for simplicity damping and fluctuation are put at one location so that the map can be simply written:

$$\begin{aligned} \overline{\varphi} &= \varphi - \frac{2\pi}{h}\alpha_\delta \delta \\ \widetilde{\delta} &= \delta + \frac{V}{E_0}\sin\left(\overline{\varphi}\right) \\ \overline{\delta} &= \lambda_\delta \widetilde{\delta} \; + \; \Delta \; . \end{aligned} \tag{1}$$

Here the quantity Δ is a stochastic variable of vanishing average obeying some distribution. λ is a number slightly below unity providing the necessary damping. And finally, α_δ is the δ-dependent momentum compaction:

$$\alpha_\delta \; = \; \sum_{n=1} \alpha_i \delta^i \; . \tag{2}$$

The other quantities are the harmonic number, the voltage and reference energy. The reader should not be distracted by the adequacy or lack of adequacy of this map. Rather it is just an example which happens to be more or less representative of a realistic situation in one degree of freedom. It will also us to display reproducable results and compare methods. We will first study the linear deterministic map. This will set the ground for the beam envelope and the Chao method.

A. The Linear Deterministic Map

We first linearize the map of Eq. (1):

$$\begin{aligned} \overline{\varphi} &= \varphi - \frac{2\pi}{h}\alpha_1 \delta \\ \widetilde{\delta} &= \delta + \frac{V}{E_0}\overline{\varphi} \\ \overline{\delta} &= \lambda \widetilde{\delta} \; + \; \Delta \; . \end{aligned} \tag{3}$$

This map can be rewritten in matrix form:

$$\begin{aligned} \overline{\mathbf{z}} &= M\mathbf{z} + \mathbf{\Delta} \\ \text{where } M &= \begin{pmatrix} 1 & -\frac{2\pi}{h}\alpha_1 \\ \frac{\lambda V}{E_0} & \lambda\left(1+\frac{2\pi V}{hE_0}\right) \end{pmatrix} . \end{aligned} \tag{4}$$

In one degree of freedom, we can rewrite the matrix M as the product of a constant times a symplectic matrix K:

$$\begin{aligned} M &= \Lambda K \\ K &= \begin{pmatrix} \lambda^{-1/2} & -\lambda^{-1/2}\frac{2\pi}{h}\alpha_1 \\ \lambda^{1/2}\frac{V}{E_0} & \lambda^{1/2}\left(1+\frac{2\pi V}{hE_0}\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos\,\mu \; + \; \alpha\sin\,\mu & \beta\sin\,\mu \\ -\gamma\sin\,\mu & \cos\,\mu \; - \; \alpha\sin\,\mu \end{pmatrix} \\ \Lambda &= \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix} . \end{aligned} \tag{5}$$

In Eq. (5), the parameters α, β, and γ are the usual Twiss parameters.

Finally, the map can be written in a normal form using the usual Courant-Snyder transformation:

$$\begin{aligned} M &= \Lambda K = A \underbrace{\Lambda R}_{N} A^{-1} \\ M &= ANA^{-1} \end{aligned} \tag{6}$$

$$\begin{aligned} \text{where } R &= \begin{pmatrix} \cos\,\mu & \sin\,\mu \\ -\sin\,\mu & \cos\,\mu \end{pmatrix} \\ \text{and } A &= \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} . \end{aligned} \tag{7}$$

These results may appear restricted to the one degree of freedom case, however in the form of Eq. (6) they generalize to the multidimensional map. In fact they can even be extended (and have been extended) to the nonlinear regime. We will explain this later as it is relevant to the main result of this paper.

B. The Deterministic Moment Map

The linear moment map is simply the matrix M:

$$\langle \overline{\mathbf{z}} \rangle \; = \; \langle M\mathbf{z} \rangle = M \langle \mathbf{z} \rangle \tag{8}$$

The quadratic moments are given by the map:

$$\begin{aligned} \langle \overline{z}_a \overline{z}_b \rangle &= \left\langle \sum_{i,j} M_{ai} M_{bj} z_i z_j \right\rangle \\ &= \sum_{i,j} M_{ai} M_{bj} \langle z_i z_j \rangle \end{aligned} \tag{9}$$

If we call the vector of moments $\mathbf{\Sigma}$, then this vector is transformed also by a matrix

$$\begin{aligned} \overline{\mathbf{\Sigma}} &= \mathbf{M}\mathbf{\Sigma} \\ &\Rightarrow \mathbf{\Sigma} = \left(\langle \mathrm{z}_1 \rangle , \langle \mathrm{z}_2 \rangle , \left\langle \mathrm{z}_1^2 \right\rangle , \left\langle \mathrm{z}_2^2 \right\rangle , \langle \mathrm{z}_1 \mathrm{z}_2 \rangle \right) \end{aligned} \tag{10}$$

The matrix for the moment, Eq. (10), is constructed from Eqs. (8) and (9).

C. The Linear Stochastic Moment Map

The stochastic part of the map is the additive term in Eq. (1):

$$\overline{\delta} \;=\; \delta \;+\; \Delta \tag{11}$$

More generally we can write a vector equation since all the phase space variables might fluctuate:

$$\overline{\mathbf{z}} \;=\; \mathbf{z} + \mathbf{\Delta} \tag{12}$$

Let us look at the change of an arbitrary moment $\langle \overline{z}_1^{m_1} \overline{z}_2^{m_2} \rangle$ under the effect of the stochastic kick:

$$\begin{aligned} \langle \overline{z}_1^{m_1} \overline{z}_2^{m_2} \rangle &= \langle (z_1 + \Delta_1)^{m_1} (z_2 + \Delta_2)^{m_2} \rangle \\ &= \sum_{a,b} \binom{m_1}{a} \binom{m_2}{b} \left\langle \Delta_1^{m_1 - a} \Delta_2^{m_2 b} \right\rangle \langle z_1^a z_2^b \rangle \end{aligned} \tag{13}$$

This stochastic map contains a matrix term and a translation term in the space of moments. We therefore rewrite as follows:

$$\begin{aligned} \overline{\mathbf{\Sigma}} &= \mathbf{S\Sigma} + \mathbf{\Theta} \\ \mathbf{S}_{ij\;ab} &= \binom{m_1}{a} \binom{m_2}{b} \left\langle \Delta_1^{m_1 - a} \Delta_2^{m_2 b} \right\rangle \\ \text{and } \Theta_{ab} &= \left\langle \Delta_1^a \Delta_2^b \right\rangle \end{aligned} \tag{14}$$

The results of Eq. (13) are quite general. Incidentally, if the stochastic process depends on phase space itself, as it usually does, then one can expand the stochastic kick in powers of the transverse variables as well. For our simple example map, the vector $\mathbf{\Theta}$ has only one non zero component:

$$\Theta_{02} \;=\; \left\langle \Delta_2^2 \right\rangle = \left\langle \Delta^2 \right\rangle \tag{15}$$

and consequently, the matrix $\mathbf{S}$ is identity to second order in the moment. For the linearized beam envelope theory, it is usually true that $\mathbf{S}$ is identity; this results holds in the case of the full six dimensional phase space.

D. Solution for the Equilibrium Moments

Because the stochastic kick Δ averages to zero, the linear moments transform completely under the effect of the deterministic matrix M and thus collapse to the origin. We are left with the quadratic moments. Combining the results of the previous section, we have:

$$\langle \overline{z}_a \overline{z}_b \rangle \;=\; \sum_{i,j} M_{ai} M_{bj} \langle z_i z_j \rangle + \left\langle \Delta^2 \right\rangle \delta_{a2} \delta_{b2} \tag{16}$$

We then change to the normalized variables as defined by Eq. (7):

$$\left\langle \overline{\zeta}_a \overline{\zeta}_b \right\rangle \;=\; \sum_{i,j} N_{ai} N_{bj} \langle \zeta_i \zeta_j \rangle + \beta \left\langle \Delta^2 \right\rangle \delta_{a2} \delta_{b2} \tag{17}$$

To simplify further Eq. (17), we introduce the so-called resonance basis (or phasors):

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \;=\; \underbrace{\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}_{B} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{18}$$

In this basis, the map is diagonal and is given by:

$$\begin{aligned} C &= \begin{pmatrix} e^{-i\mu - d} & 0 \\ 0 & e^{i\mu - d} \end{pmatrix} = B^{-1} N B \\ \text{where } \lambda &= e^{2d} \,. \end{aligned} \tag{19}$$

In the resonance basis, the full stochastic map becomes:

$$\begin{aligned} \left\langle \overline{\xi}_1^2 \right\rangle &= e^{-2i\mu - 2d} \langle \xi_1^2 \rangle - \beta \left\langle \Delta^2 \right\rangle \\ \left\langle \overline{\xi}_2^2 \right\rangle &= e^{2i\mu - 2d} \langle \xi_2^2 \rangle - \beta \left\langle \Delta^2 \right\rangle \\ \left\langle \overline{\xi}_1 \overline{\xi}_2 \right\rangle &= e^{-2d} \langle \xi_1 \xi_2 \rangle + \beta \left\langle \Delta^2 \right\rangle \end{aligned} \tag{20}$$

The equilibrium beam sizes are obtained by equating the final moments to the initial moments in Eq. (20). In resonance basis, three terms emerge:

$$\begin{aligned} \langle \xi_1^2 \rangle_\infty &= \frac{-\beta \left\langle \Delta^2 \right\rangle}{1 - e^{-2i\mu - 2d}} \\ \langle \xi_2^2 \rangle_\infty &= \frac{-\beta \left\langle \Delta^2 \right\rangle}{1 - e^{2i\mu - 2d}} \\ \langle \xi_1 \xi_2 \rangle_\infty &= \frac{\beta \left\langle \Delta^2 \right\rangle}{1 - e^{-2d}} \end{aligned} \tag{21}$$

By multiplying the matrices A and B, one can get the moments of the original phase space variables:

$$AB \;=\; \frac{1}{2} \begin{pmatrix} \sqrt{\beta} & \sqrt{\beta} \\ \frac{-\alpha - i}{\sqrt{\beta}} & \frac{-\alpha + i}{\sqrt{\beta}} \end{pmatrix} \tag{22}$$

Using this, we obtain for example,

$$\begin{aligned} \langle z_1^2 \rangle_\infty &= \frac{1}{4} \beta \left(\langle \xi_1^2 \rangle_\infty + \langle \xi_2^2 \rangle_\infty \right) + \frac{\beta}{2} \langle \xi_1 \xi_2 \rangle_\infty \\ &= -\frac{1}{2} \beta^2 \left\langle \Delta^2 \right\rangle \; Re \left\{ \frac{1}{1 - e^{i2\mu - 2d}} \right\} + \frac{\beta^2 \left\langle \Delta^2 \right\rangle}{2 \left(1 - e^{-2d} \right)} \end{aligned} \tag{23}$$

E. The Chao-Sands Approximation

The results of Sec. (II D) are actually exact for a linear system. In general, one can compute a full one turn map for the 21 moments of the six dimensional phase space of an accelerator. This map will have a deterministic part and translational part: the features in Eq. (14) are generic to the full linear and even nonlinear map. The code SAD of KEK and the library PTC use this approach for calculation of equilibrium distributions.

The reader familiar with synchrotron integral theory will notice the absence of the so-called "equilibrium emmitance." In fact, perhaps the reader has notices that, in Eq. (23), the equilibrium value of $\left\langle z_1^2\right\rangle$ depends potentially on three quantities: $\left\langle \xi_1^2\right\rangle_\infty$, $\left\langle \xi_2^2\right\rangle_\infty$ and $\left\langle \xi_1\xi_2\right\rangle_\infty$.

How do we regain the "synchrotron" integral theory? First let us regain it through mindless approximation on equation Eq. (23), keeping in mind that similar manipulations are possible on the other two moments. We first notice that the usual resonance denominators multiplying the driving terms of the 2μ resonance, namely multiplying the ξ_1^2 and ξ_2^2 phasors. When the damping decrement d is small, more exactly when $2\mu - 2d$ is far from 2π or zero, then the resonance terms are small compare to the term multiplying the $\xi_1\xi_2$. In the case, the equilibrium beam size reduces to:

$$\begin{aligned} \left\langle z_1^2\right\rangle_\infty &\approxeq \frac{\beta^2\left\langle \Delta^2\right\rangle}{2\left(1-e^{-2d}\right)} \\ &\approx \frac{\beta}{2} \underbrace{\frac{\beta\left\langle \Delta^2\right\rangle}{2d}}_{\text{equilibrium emittance}} \end{aligned} \quad (24)$$

The equilibrium emittance is the (approximate) average of $\xi_1\xi_2$ which is simply the average of the quadratic invariant of the symplectic map. In one degree of freedom, this is the Courant-Snyder invariant. The beam sizes are related to the equilibrium emittance through the usual formulas connecting the normalized variables to the original phase space variables, i.e., through the map A.

Of course, it is possible to guess these results prior to any exact moment calculations. Historically, the approximations of Sands and later that of Chao were not derived from an exact beam envelope calculation. They postulated that the equilibrium distribution level curves match the invariant curves of the original symplectic map, that is to say, the curves one derives by normalizing the one turn symplectic matrix. If the damping is very small, most particles make many rotations in the neighborhood of their symplectic trajectories before the effect of the stochasticity is appreciable. Hence it suffices to compute the equilibrium value of the function $\xi_1\xi_2$ to obtain all the moments. The formalisms resulting from this approximation are not necessarily easier to implement in a tracking code, particularily with the advent of automatic differentiation packages and modern languages with operator overloading. However they lead to simpler analytical formulas which can help the designer of a ring.

As we already mention, the Sands approximation exploits, in addition to the small damping decrement, the small longitudinal tune to rewrite the formulas in terms of expressions which are strictly well defined only in the cavity-less ring. It is not possible to illustrate this with a one-degree of freedom example. It is, for the purpose of this paper, irrelevant.

In the next section, we will jump to nonlinear calculations and illustrate how a nonlinear stochastic map diverges.

III. THE NONLINEAR PROBLEM

The general nonlinear problem can be written as

$$\overline{\mathbf{\Sigma}} = \mathbf{SM\Sigma} + \mathbf{\Theta} \quad (25)$$

where $\mathbf{M}$ is a deterministic map. This factorization is not unique in the general case. But, in accelerators, because damping is small and the Chao-Sands approximation is not bad, it is fruitful to factor the map in that manner. The fixed point of this map is given by the formula:

$$\mathbf{\Sigma}_\infty = \left(1-\mathbf{SM}\right)^{-1}\mathbf{\Theta} \quad (26)$$

Let us assume that this expression can be expanded, i.e.,

$$\mathbf{\Sigma}_\infty = \sum_{n=0}^{\infty}\left(\mathbf{SM}\right)^{\mathbf{n}}\mathbf{\Theta} \quad (27)$$

The reader can check that Eq. (27) is equivalent to starting with a Dirac delta function at the origin, that is to say, all the particles are at $\mathbf{z} = 0$ and letting it evolve until it reaches equilibrium. Remarkably, as we will see, this series often converges. We examine in the next section the reasons why this series may not converge even when the final distribution seems free of any pathologies when tracked with brute force.

A. The Perfect Integrable Deterministic Map

In this paper we concentrate on nonlinear maps which are free of resonance around the origin. Therefore let us assume that it is possible, at least to some order in the Taylor Series, to rewrite the deterministic map in a normal form reminiscent of the symplectic normal form:

$$\mathbf{m} = \mathbf{a}\circ\mathbf{n}\circ\mathbf{a}^{-1} \quad (28)$$

The normal form n is best understood through its Lie operator representation. Its Lie map $\mathcal{N}$ has the form

$$\mathcal{N} = \exp\left(\sum_{n=0}^{\infty}\mu_n J^n\partial_\phi + \sum_{n=0}^{\infty} d_n 2J^{n+1}\partial_J\right) \quad (29)$$
$$\text{where } \begin{cases}\xi_1 = \sqrt{2J}\exp\left(-i\phi\right)\\ \xi_2 = -\sqrt{2J}\exp\left(i\phi\right)\end{cases}$$

The first term in Eq. (29) represent the tune shifts with amplitude. For small damping, these terms are not appreciably different from the their value in the absence of damping. The second term contains linear damping and amplitude dependent damping. In the case of a deterministic map, the moment map is dual to the Lie map. This means that if we construct the matrix for the Lie map using the monomials in (ξ_1,ξ_2) as an expansion basis for arbitrary functions, then this matrix will be the transposed of the moment matrix. Therefore the study of the map $\mathcal{N}$ is equivalent to the study of the moment map $\mathbf{N}$ associated to the original phase space map $\mathbf{n}$ of Eq. (28).

In the case of the symplectic map, it is not possibl to removefrom $\mathcal{N}$ the tune shift terms completely. In fact, by canonical transformations, it is not possible to remove any of the symplectic terms: they are invariant under canonical perturbation. In the case of the nonsymplectic map, it is possible all the tune shifts:

$$\mathbf{m} \;=\; \mathbf{a} \circ \mathbf{b} \circ \mathbf{r} \circ \mathbf{b}^{-1} \circ \mathbf{a}^{-1} \tag{30}$$

The map $\mathbf{r}$ is simply the linear map encountered before: a rotation times a pure damping. This is the map represented by the matrix N of Eq. (5). The Lie maps associated to $\mathbf{r}$ is simply:

$$\mathcal{R} \;=\; \exp\left(\mu_0 \partial_\phi + d_0 2J\partial_J\right) \tag{31}$$

Discussion

We can ask the following question. Given the map $\mathcal{N}$ of Eq. (29) and assuming it is represents well the original map in the neighborhood of the origin which is under investigation, will the map $\mathcal{R}$ be an equally good representation? The answer answer to this is yes provided the function multiplying ∂_J in Eq. (29) does not vanish for some values of J. If it does vanish, we then have a limit cycle and potentially a reversal from damping to antidamping beyond the limit cycle. This is not possible in the case of the map of Eq. (31).

So we conclude from Eq. (31), that near the origin of phase space in an accelerator with classical rediation, all the moments collapse to zero. The map truly damps towards the origin. It should be pointed out that without radiation, the normal form predicts that a distribution whose level curves are made of the invariant J will stay invariant. The moment map however contains secular terms which makes useless once expanded: truly the damping cures this problem.

The next question we address is what happens in the presence of stochastic fluctuations to the full nonlinear moment map.

B. The Stochastic Moment Map Divergence

Naively we may expect that if the motion is a sink for the deterministic part of the map, then with a bit of luck, the results of the previous section will follow. We have found on examples that this is not necessarily the case. Indeed for some values of the nonlinearities, the series of Eq. (27) does converge and the results agree with a Monte Carlo simulation beautifully. For other values of the map, eventhough we are in a regime void of resonances, the series diverges miserably. What is the cause of this divergence? It turns out that the cause is related to secular terms in the stochastic moment map. In the case of small damping, where the Chao-Sands approximation holds, these secular terms are just the tune shift terms of Eq. (28).

Perhaps the following pseudo-physical explanation may help. In the case of a deterministic map with damping, a particle at a certain amplitude J will have secular terms in the expansion of its motion has a Taylor series. And therefore, so will have the moments. However such a particle never stays at this value of J but inexorably falls torwards the origin. Therefore eventually the secular terms tend towards zero at an exponential rate: the power of the moment map converges towards the zero matrix. In the stochastic case, the stochastic part of the matrix will fight this tendency. Indeed this is why we have an equilibrium distribution. These terms may eventually make the series once more divergent. However we cannot conclude that a divergent series means necessarily the absence of an equilibrium distribution. Just as in the symplectic case, this could be an artifact of the expansion. As we said above, numerical Monte Carlo studies confirm the existence of an equilibrim distribution.

Here we would like an exactly solvable case which will show to the reader explicitely the destructive effects on secular terms in the moment map and how their removal may restore convergence of the series of Eq. (27). To do this we will construct a very special stochastic map and analyse its properties.

Let us construct a nonlinear map whose stochastic properties are trivial thanks to symmetries. Consider again the map of Eq. (28) and let us add to it a very special stochastic force:

$$\begin{aligned}
\mathbf{m} \;&=\; \mathbf{a} \circ \mathbf{s} \circ \mathbf{n} \circ \mathbf{a}^{-1} \\
s_1(\zeta_1,\zeta_2) \;&=\; \zeta_1 + \Delta_1 \\
s_2(\zeta_1,\zeta_2) \;&=\; \zeta_2 + \Delta_2 \\
\mathbf{n} \;&=\; \lambda \circ \mathbf{r} \\
r_1 \;&=\; \cos\left(\widetilde{\mu}\right)\zeta_1 + \sin\left(\widetilde{\mu}\right)\zeta_2 \\
r_2 \;&=\; \cos\left(\widetilde{\mu}\right)\zeta_2 - \sin\left(\widetilde{\mu}\right)\zeta_1 \\
\widetilde{\mu} \;&=\; \mu + \mu'(\zeta_1^2 + \zeta_2^2) \\
\lambda(\zeta_1,\zeta_2) \;&=\; (\lambda\zeta_1, \lambda\zeta_2)\;;\; \lambda = e^{-d}.
\end{aligned} \tag{32}$$

Let assume furthermore that the stochastic variables Δ_1 and Δ_2 are totally uncorrelated but have totally identical distribution with vashing odd moments. Then the map of Eq. (32) has some remarkable properties:

1. Any initial distribution tends towards the equilibrium distribution for positive damping decrement.

2. All such distributions sit on level curves corresponding to the invariant curves of the deterministic map. In the normalized space, these are circles. In fact, the stability analysis does not depend on the distorsion map $\mathbf{a}$.

3. The equilibrium distribution does not depends on the tune shift with amplitude coefficient μ'

4. For large enough μ' the stochastic moment map has eigenvalues greater than unity and therefore violates the fundamental property of item 1.

Since the map **a** of our example is irrelevant to the discussion, we will work and state everything in normalized space. The above assertions are based on the high degree of symmetry of both the deterministic map and the stochastic force. Although we cannot easily guess the exact density of each level curve of the equilibrium distribution, the symmetry tells us that these are circles. Furthermore, it is clear that the tune shift with amplitude plays no role since its moves a given particle along a curve on which the stochastic properties are totally preserve. What is less clear is the role of the tune shift in making the stochastic moment map diverge. To see how this happens, let us compute the moment map to quartic order.

The reader will also notice that all odd moments vanish and thus we can ignore them. We will only consider distributions with even moments to facilitate the illustration. Let us compute the deterministic moment map in resonance basis. We start with the map for $\xi_1(=\zeta_1+i\zeta_2)$:

$$\begin{aligned}
\overline{\xi}_1 &= \lambda e^{-i\mu}\left(1-i\mu'\xi_1\xi_2\right)\xi_1+\cdots \\
&\Downarrow \\
\left\langle\overline{\xi}_1^2\right\rangle &= \lambda^2 e^{-i2\mu}\left(\left\langle\xi_1^2\right\rangle - i2\mu'\left\langle\xi_1^3\xi_2\right\rangle\right)+\cdots \\
\left\langle\overline{\xi}_1^3\overline{\xi}_2\right\rangle &= \lambda^4 e^{-i2\mu}\left\langle\xi_1^3\xi_2\right\rangle\cdots \\
\left\langle\overline{\xi}_1\overline{\xi}_2\right\rangle &= \lambda^2\left\langle\xi_1\xi_2\right\rangle+\cdots
\end{aligned} \tag{33}$$

We will see that the three moments (and their complex conjugate) suffice to understand the basic features of our example map. Now using Eq. (33), we write the matrix $\mathbf{M}$ for the three moments:

$$\begin{pmatrix}\left\langle\overline{\xi}_1^2\right\rangle \\ \left\langle\overline{\xi}_1^3\overline{\xi}_2\right\rangle \\ \left\langle\overline{\xi}_1\overline{\xi}_2\right\rangle\end{pmatrix} = \mathbf{M}\begin{pmatrix}\left\langle\xi_1^2\right\rangle \\ \left\langle\xi_1^3\xi_2\right\rangle \\ \left\langle\xi_1\xi_2\right\rangle\end{pmatrix}$$

$$= \begin{pmatrix}\Gamma^2 & -\Gamma^2 i2\mu' & 0 \\ 0 & \lambda^2\Gamma^2 & 0 \\ 0 & 0 & \lambda^2\end{pmatrix}\begin{pmatrix}\left\langle\xi_1^2\right\rangle \\ \left\langle\xi_1^3\xi_2\right\rangle \\ \left\langle\xi_1\xi_2\right\rangle\end{pmatrix} \tag{34}$$

$$\text{where } \Gamma = \lambda e^{-i\mu}.$$

We now produce the map for the fluctuation $\Delta(=\Delta_1+i\Delta_2)$. For a little generality, let us assume:

$$\begin{aligned}
\left\langle\Delta_1^2\right\rangle = \delta^2 + D_2/2 \quad &\text{and} \quad \left\langle\Delta_2^2\right\rangle = \delta^2 - D_2/2 \\
\left\langle\Delta_1^4\right\rangle = \delta^4 + D_4/2 \quad &\text{and} \quad \left\langle\Delta_2^4\right\rangle = \delta^4 - D_4/2\,.
\end{aligned} \tag{35}$$

The asymetries D_2 and D_4 are zero for our special map. Here we keep them different from zero just to what can happen in a more general case. We can now compute the stochastic map on the three relevant moments:

$$\begin{aligned}
\left\langle\overline{\xi}_1^2\right\rangle &= \left\langle\xi_1^2\right\rangle + \left\langle\Delta^2\right\rangle + i2\underbrace{\left\langle\xi_1\Delta\right\rangle}_{=0} \\
&= \left\langle\xi_1^2\right\rangle + \left\langle\Delta_1^2-\Delta_2^2+i2\Delta_1\Delta_2\right\rangle \\
&= \left\langle\xi_1^2\right\rangle + D_2
\end{aligned} \tag{36}$$

$$\begin{aligned}
\left\langle\overline{\xi}_1\overline{\xi}_2\right\rangle &= \left\langle\xi_1\xi_2\right\rangle + \left\langle\Delta\Delta^\star\right\rangle + i2\underbrace{\left\langle\xi_1\Delta^\star+\xi_2\Delta\right\rangle}_{=0} \\
&= \left\langle\xi_1\xi_2\right\rangle + 2\delta^2
\end{aligned} \tag{37}$$

And, finally,

$$\left\langle\overline{\xi}_1^3\overline{\xi}_2\right\rangle = \left\langle\xi_1^3\xi_2\right\rangle + 3\left\langle\xi_1\xi_2\right\rangle\Delta_2 + 6\left\langle\xi_1^2\right\rangle\delta^2 + D_4 \tag{38}$$

The stochastic kick map, as parametrized in Eq. (14), is just:

$$\begin{aligned}
\mathbf{\Theta} &= \left(D_2, D_4, 2\delta^2\right) \\
\mathbf{S} &= \begin{pmatrix}1 & 0 & 0 \\ 6\delta^2 & 1 & 3D_2 \\ 0 & 0 & 1\end{pmatrix}
\end{aligned} \tag{39}$$

Thus the full matrix of the map, including the deterministic part, is given by:

$$\mathbf{SM} = \begin{pmatrix}\Gamma^2 & -\Gamma^2 i2\mu' & 0 \\ 6\delta^2\Gamma^2 & \Gamma^2\left(\lambda^2-i12\delta^2\mu'\right) & 3D_2\lambda^2 \\ 0 & 0 & \lambda^2\end{pmatrix} \tag{40}$$

Looking at this simple example, we notice that the equilibrium value of $\langle\xi_1\xi_2\rangle$ does not depend on the asymetrical terms D_2 and D_4:

$$\Sigma_3^\infty = \left\langle\xi_1\xi_2\right\rangle_\infty = \frac{2\delta^2}{1-\lambda^2} \tag{41}$$

One can then compute the equilibrium values of the two other moments by solving:

$$\begin{aligned}
\left(\Gamma^2-1\right)\Sigma_1^\infty - \Gamma^2 i2\mu'\Sigma_2^\infty &= -D_2 \\
6\delta^2\Gamma^2\Sigma_1^\infty + \left\{\Gamma^2\left(\lambda^2-i12\delta^2\mu'\right)-1\right\}\Sigma_2^\infty &= -D_4 \\
&\quad -\frac{6D_2\delta^2\lambda^2}{1-\lambda^2}
\end{aligned} \tag{42}$$

There are two interesting thing to extract out of this example. First of all, we can confirm that in the limit of small damping the equilibrium values Σ_1^∞ and Σ_2^∞ are small compared to the equilibrium "emittance" Σ_3^∞. This is confirm by the determinant of the system of Eq. (42):

$$\text{Det} = \left(1-\Gamma^2\right)\left(1-\Gamma^2\lambda^2\right) + i12\delta^2\mu' \tag{43}$$

This particular function, unlike the denominator in Eq. (41), is generally a number with a modulus much larger than the damping decrement d. One notices first a product of two phasors term and a nonlinear dephasing term related to the tune shift with amplitude μ'. This seems to indicate that a moderate amount of tune shift with amplitude will actually detune us away and enhance the effect of the emittance

However, we can verify that an exces of tune shift with amplitude will render this particular calculation meaningless. We know if we select a perfectly symmetrically circular stochastic process, i.e., $D_2 = 0$ and $D_4 = 0$, then all initial distribution should tend towards the equilibrium distribution correspounding to $\mu' = 0$. This can be checked numerically and even demonstrated for a large class of stochastic terms using Brower's fixed point theorem. If we go back to the matrix **SM** and compute the eigenvalues of the matrix controlling the motion of the first two planes. They are given by the formula:

$$\lambda_\pm \;=\; \Gamma^2 \left\{ \frac{\Omega \pm \sqrt{\Omega^2 - 4\lambda^2}}{2} \right\} \tag{44}$$
$$\text{where } \Omega \;=\; \lambda^2 + 1 - i12\delta^2\mu'$$

For $\mu' = 0$, the two eigenvalues correspound to the phasors eigenvalues, i.e., $\Gamma^2\lambda^2$ and Γ^2; thus all moments converge towards a the equilibrium moments. However for a given value of μ', some distribution which are not purely along the emittance direction will diverge. This is totally unphysical in this symmetrical case. Unfortunately it is unphysical in many symmetrical situation as well.

To check the above assertions, we tried the following example:

$$\begin{aligned} \mu &= 0.39\ 2\pi \\ d &= 10^{-4} \\ \delta^2 &= 8\ 10^{-10} \\ D_2 &= -\delta^2 \\ D_4 &= -\delta^4 \end{aligned} \tag{45}$$

These for the numerical simulation we chose a very simple random variable, namely the variable Δ_1 is always zero and the variable Δ_2 takes the value $\pm\sqrt{2}\delta$ with an equal probability. According to our analysis, the stochastic map, if computed to 4$^{\text{th}}$ order, should be unstable for a value of the tune shift parameter somewhere in the range

$$2.8132 < \frac{\mu'}{2\pi} < 2.8136 \quad .$$

One can perform either with a Taylor series package or an algebraic manipulator the full calculation. In our case we implemented this stochastic map to arbitrary order using our FPP package. The results, if carried to 4$^{\text{th}}$ diverge precisely in the predicted range. Of course, it turns out that higher order calculations diverge for even smaller values of the tune shift parameter. However, tracking shows the system converges towards an equilibrium even for ludicrous values of the amplitude tune shift ($> 1,000,000!$) and that this equilibrium is nearly independent of this parameters. In the next we draw some conclusion from this example and propose some palliative to the unphysical divergence of the moment map.

IV. RESUSCITATING THE CHAO-SANDS APPROXIMATION

In the previous section we showed, through an example, that the tune shift with amplitude can destroy completely the beam envelope calculation. This should not be a surprise since in the case of a pure deterministic map, it leads to the same disastrous result. Tune shifts with amplitude and filamentation are one and the same thing. These are secular terms in the map which should never be expanded.

Let us now assume a map for which we find, through tracking perhaps, that the equilibrium distribution level curves sit very nearly on the invariant of the nearby radiation-free symplectic map. Generally, unless the map is a constructed pathology as in Sec. (III), such a map must have a small damping compared the linear reasonance terms.

On such a system, any distribution which is a function of the symplectic invariants will evolve into an equivalent distribution. Furthermore, thanks to both filamentation and stochasticity, any distribution which is not a function of the invariants alone, will evolve very rapidly towards such a distribution. In fact, if we look at ergodic averages over a few turns, then the progression is even faster. For example, imagine a small blob of particles: it will rotate in phase space along the symplectic trajectories so that in a few turns the time average of the distribution will correspound to a distribution uniformally spreaded in normalized phase. Moreover, in slightly longer time, still small compare to a damping time, the tune shifts with amplitude with filament the distribution so that for practical purposes it will be uniform in normalized phase.

Suppose that for this particular map, we find that the expansion as given by Eq. (27) diverges despite a well behaved distribution in Monte-Carlo simulation. What can we do about the moment calculation? Let us rewrite Eq. (26) as follows:

$$\mathbf{\Sigma}_\infty - \mathbf{SMB^{-1}B\Sigma}\infty \;=\; \mathbf{\Theta} \tag{46}$$

Let us assume that the map B commutes with the original symplectic map, i.e., it has the same invariants. In fact, if the original symplectic Lie map $\mathcal{M}_s$ can be normalized as

$$\mathcal{M}_s \;=\; \mathcal{A}^{-1}\mathcal{R}\mathcal{A} \tag{47}$$

then we can assume that the Lie map for B can be written as

$$\mathcal{B} \;=\; \mathcal{A}^{-1}\mathcal{R}_b\mathcal{A} \tag{48}$$

where $\mathcal{R}_b$ is an amplitude dependent rotation. Now, under the Chao-Sand assumption, we conclude that

$$\mathbf{B\Sigma}_\infty \;=\; \mathbf{\Sigma}_\infty + \cdots \mathrm{O(damping)} \tag{49}$$

This means that Eq. (46) reduces to

$$\begin{aligned}\boldsymbol{\Sigma}_\infty &= \left\{1-\mathbf{SMB^{-1}}\right\}^{-1}\boldsymbol{\Theta} \\ &= \sum_{n=0}^{\infty}\left\{\mathbf{SMB^{-1}}\right\}^{\mathbf{n}}\boldsymbol{\Theta} \end{aligned} \qquad (50)$$

In the previous example of Sec. (III), the equality in Eq. (49) was actually exact for vanishing D_2 and D_4. Now here comes the application of the Chao-Sands trick. In cases when the map **SM** contains eigenvalues bigger than one and thus whose iteration becomes potentially unphysical, we may select the map **B** to annihilate most of the tune shift in the original stochastic map. Thus the map of Eq. (50) is likely to converge unlike that of Eq. (27). Furthermore, if we restrict ourselves to distributions sitting on invariant, it also describes their time evolution. The dimension of this subspace is very tiny in the space of moments since it is in one to one correspondance with the space of nonlinear rotation. Thus, for example, if we look at moments up to order 4, in one degree of freedom, then it is a space describe by two parameters. However for maps where the Chao-Sands approximation is valid, the dynamics of any distribution converges very rapidly towards a dynamical restricted to this small subspace. In fact, in the linear case, it is remarkable in the transverse plane, even the constant term Θ is often aligned very well along the transverse invariants, i.e., the so-called H functions of the Sands theory. This is because machines are transversely highly periodic (made out of cells) and therefore the one-term stochastic map is already the a power of a primitive cell. In the Berkeley Advance light Source one can check that the transverse fluctuation part of Θ is indeed along the H-function while the longitudinal part is not. Obviously, because there is only one cavity in the ring, the one turn longitudinal map is not at all along the H-function. However, the small damping projects ultimately everything along the invariants: Chao-Sands theory just works fine!

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[1] Sands formalism takes advantage of the asymetry between the transverse and longitudinal dynamics, namely, $\nu_L \ll \nu_x, \nu_y$. This permits to derive convenient formulas in terms of quantities well-defined only, strictly speaking, in the cavity-less ring, such as closed orbit dispersion. For the purpose of this paper, we prefer to confuse it with the more symmetric Chao formalism.