DESY 95–189 (1995)

Construction of Nonlinear Symplectic Six-Dimensional Thin - Lens Maps by Exponentiation

K. Heinemann, G. Ripken, F. Schmidt*

October 25, 1995

Abstract

The aim of this paper is to construct six - dimensional symplectic thin–lens transport maps for the tracking program SIXTRACK [2], continuing an earlier report [1] by using another method which consistes in applying Lie series and exponentiation as described by W. Gröbner [3] and for canonical systems by A.J. Dragt [4]. As in Ref. [1] we firstly use an approximate Hamiltonian obtained by a series expansion of the square root

$$\left\{1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2}\right\}^{1/2}$$

up to first order in terms of the quantity

$$\frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2} .$$

Furthermore, nonlinear crossing terms due to the curvature in bending magnets are neglected. An improved Hamiltonian, excluding solenoids, is introduced in Appendix A by using the unexpanded square root mentioned above, but neglecting again nonlinear crossing terms in bending magnets. It is shown that the thin - lens maps remain unchanged and that the corrections due to the new Hamiltonian are fully absorbed into the drift spaces. Finally a symplectic treatment of the crossing terms appearing in bending magnets is presented in Appendix B, taking into account only the lowest order. The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy and shall be incorporated into the tracking code SIXTRACK [2].

*CERN, SL-Division, Geneva, Switzerland

Contents

1 Introduction **4**

2 The Canonical Equations of Motion **4**
2.1 Notation 4
2.2 The Hamiltonian in Machine Coordinates 4

3 Description of the Thin - Lens Method **7**
3.1 Thin - Lens Approximation 7
3.2 Integration by Lie - Series 12
3.2.1 General Autonomous Case 12
3.2.2 Calculation of the Thin - Lens Transport Map for the Central Region. . 15

4 Thin - Lens Approximation for Various Kinds of Magnets and for Cavities **17**
4.1 Quadrupole 17
4.1.1 Exponentiation 17
4.1.2 Thin - Lens Transport Map 18
4.2 Skew Quadrupole 19
4.2.1 Exponentiation 19
4.2.2 Thin - Lens Transport Map 20
4.3 Bending Magnet 21
4.3.1 Exponentiation 21
4.3.2 Thin - Lens Transport Map 22
4.4 Sextupole 22
4.4.1 Exponentiation 22
4.4.2 Thin - Lens Transport Map 23
4.5 Octupole 24
4.5.1 Exponentiation 24
4.5.2 Thin - Lens Transport Map 25
4.6 Synchrotron - Magnet 25
4.6.1 Exponentiation 25
4.6.2 Thin - Lens Transport Map 26
4.7 Solenoid 27
4.7.1 Exponentiation 27
4.7.2 Thin - Lens Transport Map 30
4.8 Cavity 31
4.8.1 Exponentiation 31
4.8.2 Thin - Lens Transport Map 32

5 Summary **32**

Acknowledgments **32**

Appendix A: Thin - Lens Approximation with an Improved Hamiltonian **32**
A.1 The Improved Hamiltonian 33
A.2 Equations of Motion 35
A.2.1 Drift Space 35

A.2.2 The Central Part . . . 37

Appendix B: Bending Magnet with Nonlinear Crossing Terms **41**
B.1 The Hamiltonian . . . 41
B.2 Thin - Lens Approximation . . . 42
B.2.1 The Component $\hat{\mathcal{H}}_L^{(1)}$ . . . 42
B.2.2 The Component $\hat{\mathcal{H}}_L^{(2)}$ . . . 45

Appendix C: Superposition of Solenoids and Quadrupoles **51**
C.1 Exponentiation . . . 51
C.2 Thin - Lens Transport Map . . . 56

References **57**

1 Introduction

Continuing an earlier report [1], in this paper we show how to solve the nonlinear canonical equations of motion in the framework of the fully six–dimensional formalism for various kinds of magnets (bending magnets, quadrupoles, synchrotron magnets, skew quadrupoles, sextupoles, octupoles, solenoids), using an approach different from Ref. [1], namely by applying Lie series and exponentiation to a kick approximation [3, 4] [1] .

In addition to Ref. [1] we also study the thin - lens formalism for an improved Hamiltonian which is exact outside solenoids and bending magnets. It is shown that the thin - lens maps obtained earlier remain unchanged and that the corrections due to the new Hamiltonian only appear in the drift spaces.

The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy and shall be incorporated into the tracking code SIXTRACK [2].

The paper is organized as follows:

In chapter 2 the general canonical equations of motion are derived. The thin - lens method using Lie series and exponentiation is described in chapter 3. Using the thin–lens approximation the equations of motion are solved for each element in chapter 4. The improved Hamiltonian is introduced in Appendix A. In addition to Ref. [1] the influence of nonlinear "crossing terms" resulting from the curvature in bending magnets is investigated in Appendix B and a superposition of a solenoid and a quadrupole in Appendix C. Finally a summary of the results is presented in chapter 5.

2 The Canonical Equations of Motion

2.1 Notation

The formalism and notation in this paper will be identical to that used in Ref. [1]. Thus we will begin by simply stating the canonical equations of motion already used in this earlier paper and refer the reader to the latter for details.

2.2 The Hamiltonian in Machine Coordinates

The Hamiltonian for orbital motion in storage rings reads as [1]:

$$\begin{aligned}
\mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma; s) &= p_\sigma - [1 + f(p_\sigma)] \cdot [1 + K_x \cdot x + K_z \cdot z] \times \\
&\qquad \left\{ 1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} \\
&+ \frac{1}{2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 - \frac{1}{2} \cdot g \cdot (z^2 - x^2) - N \cdot xz \\
&+ \frac{\lambda}{6} \cdot (x^3 - 3\, x z^2) \\
&+ \frac{\mu}{24} \cdot (z^4 - 6\, x^2\, z^2 + x^4) \\
&+ \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right]
\end{aligned} \qquad (2.1a)$$

[1] There is a vast literature on solving differential equations by Lie series. A nice treatment is given by Ref. [3].

with

$$f(p_\sigma) \quad = \quad \frac{1}{\beta_0}\sqrt{(1+\beta_0^2\cdot p_\sigma)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2} - 1 \tag{2.1b}$$

(g, N, H, K_x, K_z, λ, and μ are defined in Ref. [1]).

Since

$$\begin{aligned} |p_x + H\cdot z| &\ll 1\ ; \\ |p_z - H\cdot x| &\ll 1 \end{aligned}$$

the square root

$$\left[1 - \frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]^2}\right]^{1/2}$$

in (2.1) may be expanded in a series:

$$\begin{aligned} &\left[1 - \frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]^2}\right]^{1/2} = \\ &\qquad 1 - \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]^2} + \cdots . \end{aligned} \tag{2.2}$$

The power at which the series is truncated defines the order of the approximation to the particle motion.

In the following we will use (as in Ref. [1]) the approximation:

$$\begin{aligned} \mathcal{H} \quad = \quad & \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]} + \\ & p_\sigma - [1 + K_x\cdot x + K_z\cdot z]\cdot f(p_\sigma) + \\ & \frac{1}{2}[K_x^2 + g]\cdot x^2 + \frac{1}{2}[K_z^2 - g]\cdot z^2 - N\cdot x\,z + \\ & \frac{\lambda}{6}\cdot(x^3 - 3\,x z^2) + \frac{\mu}{24}\cdot(z^4 - 6\,x^2 z^2 + x^4) + \\ & \frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right] . \end{aligned} \tag{2.3}$$

An improved Hamiltonian is introduced in Appendix A.

The canonical equations corresponding to the Hamiltonian (2.3) take the form:

$$\frac{d}{ds}\,\vec{y} \quad = \quad -\underline{S}\cdot\frac{\partial\mathcal{H}}{\partial\vec{y}} \tag{2.4}$$

with

$$\vec{y}^T = (y_1,\, y_2,\, y_3,\, y_4,\, y_5,\, y_6) \equiv (x,\, p_x,\, z,\, p_z,\, \sigma,\, p_\sigma)$$

and

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \; ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \tag{2.5}$$

or, written in components:

$$\frac{d}{ds}\, x = +\frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x + H \cdot z}{[1 + f(p_\sigma)]} \; ; \tag{2.6a}$$

$$\begin{aligned} \frac{d}{ds}\, p_x &= -\frac{\partial \mathcal{H}}{\partial x} \\ &= +\frac{[p_z - H \cdot x]}{[1 + f(p_\sigma)]} \cdot H - [K_x^2 + g] \cdot x + N \cdot z + K_x \cdot f(p_\sigma) \\ &\quad -\frac{\lambda}{2} \cdot (x^2 - z^2) - \frac{\mu}{6} \cdot (x^3 - 3\,x\,z^2) \; ; \end{aligned} \tag{2.6b}$$

$$\frac{d}{ds}\, z = +\frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z - H \cdot x}{[1 + f(p_\sigma)]} \; ; \tag{2.6c}$$

$$\begin{aligned} \frac{d}{ds}\, p_z &= -\frac{\partial \mathcal{H}}{\partial z} \\ &= -\frac{[p_x + H \cdot z]}{[1 + f(p_\sigma)]} \cdot H - [K_z^2 - g] \cdot z + N \cdot x + K_z \cdot f(p_\sigma) \\ &\quad +\lambda \cdot xz - \frac{\mu}{6} \cdot (z^3 - 3\,x^2\,z) \; ; \end{aligned} \tag{2.6d}$$

$$\begin{aligned} \frac{d}{ds}\, \sigma &= +\frac{\partial \mathcal{H}}{\partial p_\sigma} \\ &= 1 - [1 + K_x \cdot x + K_z \cdot z] \cdot f'(p_\sigma) \\ &\quad -\frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2} \cdot f'(p_\sigma) \\ &= 1 - [1 + K_x \cdot x + K_z \cdot z] \cdot f'(p_\sigma) \\ &\quad -\frac{1}{2} \cdot [(x')^2 + (z')^2] \cdot f'(p_\sigma) \; ; \end{aligned} \tag{2.6e}$$

$$\frac{d}{ds}\,p_\sigma \;=\; -\frac{\partial\mathcal{H}}{\partial\sigma}$$

$$= \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\ . \qquad (2.6f)$$

In detail, one has :

a)	$K_x^2+K_z^2\neq 0;$	$g=N=\lambda=\mu=H=V=0:$	bending magnet;
b)	$g\neq 0;$	$K_x=K_z=N=\lambda=\mu=H=V=0:$	quadrupole;
c)	$N\neq 0;$	$K_x=K_z=g=\lambda=\mu=H=V=0:$	skew quadrupole;
d)	$\lambda\neq 0;$	$K_x=K_z=g=N=\mu=H=V=0:$	sextupole;
e)	$\mu\neq 0;$	$K_x=K_z=g=N=\lambda=H=V=0:$	octupole;
f)	$H\neq 0;$	$K_x=K_z=g=N=\lambda=\mu=V=0:$	solenoid;
g)	$V\neq 0;$	$K_x=K_z=g=N=\lambda=\mu=H=0:$	cavity.

3 Description of the Thin - Lens Method

3.1 Thin - Lens Approximation

The equations of motion (2.6) have the general form :

$$\frac{d}{ds}\,y_i \;=\; \vartheta_i\,(y_1,\,y_2,\,y_3,\,y_4,\,y_5,\,y_6;\,s)\ ; \qquad (3.1a)$$

$$(i=1,2,3,4,5,6)$$

or

$$\frac{d}{ds}\,\vec{y} \;=\; \vec{\vartheta}\,(\vec{y};\,s) \qquad (3.1b)$$

with

$$\vec{\vartheta}^{\,T} \;=\; (\vartheta_1,\,\vartheta_2,\,\vartheta_3,\,\vartheta_4,\,\vartheta_5,\,\vartheta_6)$$

and

$$\vartheta_1(\vec{y};s) \;=\; +\frac{p_x}{[1+f(p_\sigma)]}+\frac{H(s)\cdot z}{[1+f(p_\sigma)]}\ ; \qquad (3.2a)$$

$$\begin{aligned}\vartheta_2(\vec{y};s) \;=\;& +\frac{p_z}{[1+f(p_\sigma)]}\cdot H(s)-\frac{H(s)\cdot x}{[1+f(p_\sigma)]}\cdot H(s)+\ K_x(s)\cdot f(p_\sigma)\\ &-[K_x^2(s)+g(s)]\cdot x+N(s)\cdot z\\ &-\frac{\lambda(s)}{2}\cdot(x^2-z^2)-\frac{\mu(s)}{6}\cdot(x^3-3\,x\,z^2)\ ;\end{aligned} \qquad (3.2b)$$

$$\vartheta_3(\vec{y};s) \;=\; +\frac{p_z}{[1+f(p_\sigma)]}-\frac{H(s)\cdot x}{[1+f(p_\sigma)]}\ ; \qquad (3.2c)$$

$$\vartheta_4(\vec{y};s) = -\frac{p_x}{[1+f(p_\sigma)]} \cdot H(s) - \frac{H(s)\cdot z}{[1+f(p_\sigma)]} \cdot H(s) + K_z(s)\cdot f(p_\sigma) - [K_z(s)^2 - g(s_0)]\cdot z + N(s)\cdot x + \lambda(s)\cdot xz - \frac{\mu(s)}{6}\cdot(z^3 - 3\,x^2\,z)\ ; \tag{3.2d}$$

$$\vartheta_5(\vec{y};s) = 1 - [K_x(s)\cdot x + K_z(s)\cdot z]\cdot f'(p_\sigma) - \frac{1}{2}\cdot\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) - \frac{1}{2}\cdot\frac{H(s)^2\cdot[x^2+z^2]}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) - \frac{H(s)\cdot[p_x\cdot z - p_z\cdot x]}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma)\ ; \tag{3.2e}$$

$$\vartheta_6(\vec{y};s) = \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right]\ . \tag{3.2f}$$

Equation (3.1) represents a system of differential equations the solutions of which can be written in the form:

$$\vec{y}(s_f) = \underline{T}(s_f, s_i)\,\vec{y}(s_i) \tag{3.3}$$

by defining a transport operator $\underline{T}(s_f, s_i)$ connecting the final vector $\vec{y}(s_f)$ at position s_f with the initial vector $\vec{y}(s_i)$ at position s_i.

The aim of this chapter is now to calculate the transport map $\underline{T}(s_f, s_i)$ (approximately) by using symplectic kicks.

We achieve that in two steps:

In a first step we decompose the r.h.s. of (3.1) into two components:

$$\vec{\vartheta} = \vec{\vartheta}_D + \vec{\vartheta}_L \tag{3.4}$$

gathering in $\vec{\vartheta}_L$ all terms of $\vec{\vartheta}$ containing the external electric and magnetic fields (expressed by the lens functions $V,\ g,\ N,\ H,\ K_x,\ K_z,\ \lambda,\ \mu$).

As a result, the component $\vec{\vartheta}_D$ in (3.4) then corresponds to the Hamiltonian

$$\mathcal{H}_D = \frac{1}{2}\cdot\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]} + p_\sigma - f(p_\sigma) \tag{3.5}$$

(to be obtained from (2.3) by neglecting all external fields) leading to the (canonical) equations of motion for a pure drift space:

$$\frac{d}{ds}x = +\frac{\partial}{\partial p_x}\mathcal{H}_D(x,p_x,z,p_z,\sigma,p_\sigma) = \frac{p_x}{[1+f(p_\sigma)]}\ ; \tag{3.6a}$$

$$\frac{d}{ds}p_x = -\frac{\partial}{\partial x}\mathcal{H}_D(x,p_x,z,p_z,\sigma,p_\sigma) = 0 \quad\Longrightarrow\quad p_x = const\ ; \tag{3.6b}$$

$$\begin{aligned}
\frac{d}{ds}\,z &= +\frac{\partial}{\partial p_z}\,\mathcal{H}_D(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= \frac{p_z}{[1+f(p_\sigma)]}\ ; \qquad (3.6c)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,p_z &= -\frac{\partial}{\partial z}\,\mathcal{H}_D(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= 0 \quad\Longrightarrow\quad p_z \;=\; const\ ; \qquad (3.6d)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,\sigma &= +\frac{\partial}{\partial p_\sigma}\mathcal{H}_D(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= 1 - f'(p_\sigma) - \frac{1}{2}\cdot\left[\,(p_x)^2 + (p_z)^2\,\right]\cdot\frac{f'(p_\sigma)}{[1+f(p_\sigma)]^2}\ ; \qquad (3.6e)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,p_\sigma &= -\frac{\partial}{\partial \sigma}\,\mathcal{H}_D(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= 0 \quad\Longrightarrow\quad p_\sigma \;=\; const \qquad (3.6f)
\end{aligned}$$

(see also eqn. (2.6)). The solutions for a drift of length l are:

$$x^f \;=\; x^i + \frac{p_x^i}{[1+f(p_\sigma^i)]}\cdot l\ ; \qquad (3.7a)$$

$$p_x^f \;=\; p_x^i\ ; \qquad (3.7b)$$

$$z^f \;=\; z^i + \frac{p_z^i}{[1+f(p_\sigma^i)]}\cdot l\ ; \qquad (3.7c)$$

$$p_z^f \;=\; p_z^i\ ; \qquad (3.7d)$$

$$\sigma^f \;=\; \sigma^i + \left\{1 - f'(p_\sigma^i) - \frac{1}{2}\cdot[(p_x^i)^2 + (p_z^i)^2]\cdot\frac{f'(p_\sigma^i)}{[1+f(p_\sigma^i)]^2}\right\}\cdot l\ ; \qquad (3.7e)$$

$$p_\sigma^f \;=\; p_\sigma^i \qquad (3.7f)$$

The second component $\vec{\vartheta}_L$ corresponds to the Hamiltonian

$$\begin{aligned}
\mathcal{H}_L \;=\; & \frac{H}{[1+f(p_\sigma)]}\cdot[p_x\cdot z - p_z\cdot x] + \frac{1}{2}\cdot\frac{H^2}{[1+f(p_\sigma)]}\cdot\left[x^2+z^2\right] - \\
& -[K_x\cdot x + K_z\cdot z]\cdot f(p_\sigma) + \\
& \frac{1}{2}\,[K_x^2+g]\cdot x^2 + \frac{1}{2}\,[K_z^2-g]\cdot z^2 - N\cdot x\,z + \\
& \frac{\lambda}{6}\cdot(x^3 - 3\,x z^2) + \frac{\mu}{24}\cdot(z^4 - 6\,x^2\,z^2 + x^4) +
\end{aligned}$$

$$\frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \tag{3.8}$$

containing the remaining terms in eqn. (2.3). In particular there are no p_x^2 or p_z^2 terms.

Thus we have

$$\mathcal{H} = \mathcal{H}_L + \mathcal{H}_D$$

and

$$\begin{aligned} \vec{\vartheta}_D &= -\underline{S} \cdot \frac{\partial}{\partial \vec{y}} \mathcal{H}_D \ ; \\ \vec{\vartheta}_L &= -\underline{S} \cdot \frac{\partial}{\partial \vec{y}} \mathcal{H}_L \ , \end{aligned}$$

where the matrix $\underline{S}$ is given by eqn. (2.5).

In the second step we replace the function $\vec{\vartheta}$ in (3.4) for a thin lens of length Δs at position s_0 by [1]

$$\begin{aligned} \vec{\vartheta}_{mod}(\vec{y}; s) &= \vec{\vartheta}_D(\vec{y}) + \vec{\vartheta}_L(\vec{y}; s) \cdot \Delta s \cdot \delta(s - s_0) \\ &= \vec{\vartheta}_D(\vec{y}) + \vec{\vartheta}_L(\vec{y}; s_0) \cdot \Delta s \cdot \delta(s - s_0) \end{aligned} \tag{3.9}$$

with

$$\vec{\vartheta}_L(\vec{y}; s_0) = -\underline{S} \cdot \frac{\partial}{\partial \vec{y}} \hat{\mathcal{H}}_L \tag{3.10}$$

and

$$\begin{aligned} \hat{\mathcal{H}}_L \equiv\ & \mathcal{H}_L(\vec{y}; s_0) \\ =\ & \frac{H(s_0)}{[1 + f(p_\sigma)]} \cdot [p_x \cdot z - p_z \cdot x] + \frac{1}{2} \cdot \frac{H^2(s_0)}{[1 + f(p_\sigma)]} \cdot \left[x^2 + z^2\right] - \\ & [K_x(s_0) \cdot x + K_z(s_0) \cdot z] \cdot f(p_\sigma) + \\ & \frac{1}{2} [K_x^2(s_0) + g(s_0)] \cdot x^2 + \frac{1}{2} [K_x^2(s_0) - g(s_0)] \cdot z^2 - N(s_0) \cdot x\, z + \\ & \frac{\lambda(s_0)}{6} \cdot (x^3 - 3\, x z^2) + \frac{\mu(s_0)}{24} \cdot (z^4 - 6\, x^2\, z^2 + x^4) + \\ & \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s_0)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \ , \end{aligned} \tag{3.11}$$

whereby the new function $\vec{\vartheta}_{mod}$ in (3.9) results from the modified Hamiltonian

$$\mathcal{H}_{mod} = \mathcal{H}_D + \hat{\mathcal{H}}_L \cdot \Delta s \cdot \delta(s - s_0) \ . \tag{3.12}$$

In order to solve eqn. (3.1) using the modified function $\vec{\vartheta}_{mod}$ in (3.9), we then have to decompose the region

$$s_0 - \frac{\Delta s}{2} \le s \le s_0 + \frac{\Delta s}{2}$$

of the lens into three parts :

$$\begin{aligned}
\text{region } I &: \; s_0 - \frac{\Delta s}{2} \le s \le s_0 - \epsilon \; ; && (3.13a)\\
\text{region } II &: \; s_0 - \epsilon \le s \le s_0 + \epsilon \; ; && (3.13b)\\
\text{region } III &: \; s_0 + \epsilon \le s \le s_0 + \frac{\Delta s}{2} \; ; && (3.13c)
\end{aligned}$$

$$(0 < \epsilon \to 0) \; .$$

For region I and III we obtain a drift space of length $l = \dfrac{\Delta s}{2}$, described by the differential equation

$$\frac{d}{ds}\,\vec{y}(s) \;=\; \vec{\vartheta}_D \tag{3.14}$$

the solution of which is given by eqn. (3.7) and may be expressed by a transport operator $\underline{T}_D(l)$.

The equation of motion for the central region II reads as :

$$\frac{d}{ds}\,\vec{y}(s) \;=\; \vec{\vartheta}_L(\vec{y};s_0)\cdot\Delta s\cdot\delta(s-s_0) \tag{3.15}$$

with

$$(\vec{\vartheta}_L)^T \;=\; (\vartheta_{L1},\, \vartheta_{L2},\, \vartheta_{L3},\, \vartheta_{L4},\, \vartheta_{L5},\, \vartheta_{L6})$$

and

$$\begin{aligned}
\vartheta_{L1}(\vec{y};s_0) &= +\frac{\partial}{\partial p_x}\,\hat{\mathcal{H}}_L(x,p_x,z,p_z,\sigma,p_\sigma)\\
&= +\frac{H(s_0)\cdot z}{[1+f(p_\sigma)]}\; ; && (3.16a)
\end{aligned}$$

$$\begin{aligned}
\vartheta_{L2}(\vec{y};s_0) &= -\frac{\partial}{\partial x}\,\hat{\mathcal{H}}_L(x,p_x,z,p_z,\sigma,p_\sigma)\\
&= +\frac{p_z}{[1+f(p_\sigma)]}\cdot H(s_0) - \frac{H(s_0)\cdot x}{[1+f(p_\sigma)]}\cdot H(s_0)\\
&\quad + K_x(s_0)\cdot f(p_\sigma) - [K_x^2(s_0)+g(s_0)]\cdot x + N(s_0)\cdot z\\
&\quad - \frac{\lambda(s_0)}{2}\cdot(x^2-z^2) - \frac{\mu(s_0)}{6}\cdot(x^3-3\,x\,z^2)\; ; && (3.16b)
\end{aligned}$$

$$\begin{aligned}
\vartheta_{L3}(\vec{y};s_0) &= +\frac{\partial}{\partial p_z}\,\hat{\mathcal{H}}_L(x,p_x,z,p_z,\sigma,p_\sigma)\\
&= -\frac{H(s_0)\cdot x}{[1+f(p_\sigma)]}\; ; && (3.16c)
\end{aligned}$$

$$\vartheta_{L4}(\vec{y};s_0) = -\frac{\partial}{\partial z}\,\hat{\mathcal{H}}_L(x,p_x,z,p_z,\sigma,p_\sigma)$$

$$
\begin{aligned}
&= -\frac{p_x}{[1+f(p_\sigma)]} \cdot H(s_0) - \frac{H(s_0)\cdot z}{[1+f(p_\sigma)]} \cdot H(s_0) \\
&\quad +K_z(s_0)\cdot f(p_\sigma) - [K_z^2(s_0) - g(s_0)]\cdot z + N(s_0)\cdot x \\
&\quad +\lambda(s_0)\cdot xz - \frac{\mu(s_0)}{6}\cdot (z^3 - 3\,x^2\,z)\ ; \qquad (3.16\text{d})
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_{L5}(\vec{y};s_0) &= +\frac{\partial}{\partial p_\sigma}\hat{\mathcal{H}}_L(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= -[K_x(s_0)\cdot x + K_z(s_0)\cdot z]\cdot f'(p_\sigma) \\
&\quad -\frac{1}{2}\cdot\frac{H(s_0)^2\cdot[x^2+z^2]}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) - \frac{H(s_0)\cdot[p_x\cdot z - p_z\cdot x]}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) \qquad (3.16\text{e})
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_{L6}(\vec{y};s_0) &= -\frac{\partial}{\partial \sigma}\hat{\mathcal{H}}_L(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= \frac{1}{\beta_0^2}\cdot\frac{eV(s_0)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \qquad (3.16\text{f})
\end{aligned}
$$

(see eqns. (3.10) and (3.11)) and determines the transport map

$$
\underline{T}_L \equiv \underline{T}(s_0+0, s_0-0) \qquad (3.17)
$$

of region II.

Finally the transport map of the whole lens takes the form:

$$
\underline{T}(s_0+\Delta s/2,\ s_0-\Delta s/2) = \underline{T}_D(\Delta s/2)\cdot\underline{T}_L\cdot\underline{T}_D(\Delta s/2) \qquad (3.18)
$$

corresponding to the decomposition of the length Δs into three parts (see eqn. (3.13)).

Note that the (nonlinear) transport maps $\underline{T}_D$ and $\underline{T}_L$ corresponding to (3.14) and (3.15) and thus also $\underline{T}(s_0+\Delta s/2,\ s_0-\Delta s/2)$ in (3.18) are automatically symplectic for an arbitrary Δs due to the canonical structure of the equations of motion (see also Ref. [4] and Appendix A in Ref. [1]).

In the limit

$$
\Delta s \longrightarrow 0
$$

one obtains the exact solution of the canonical equations of motion corresponding to the starting Hamiltonian (2.3).

Since $\underline{T}_D$ is already known from eqn. (3.7) we are left with the problem of calculating the transport map $\underline{T}_L$ by solving eqn. (3.15). This is done in the next section, using a Lie series and exponentiation.

3.2 Integration by Lie - Series

3.2.1 General Autonomous Case

In the thin - lens approximation the equations to be solved are not autonomous but the s - dependence is trivial which reduces the calculation to an autonomous system.

An autonomous system of differential equations of the form:

$$\frac{d}{ds}\, y_i \;=\; \tilde{\vartheta}_i\,(y_1,\, y_2,\, \ldots,\, y_n) \;\; ; \;\; \frac{\partial}{\partial s}\, \tilde{\vartheta}_i \;=\; 0 \; ; \qquad (i = 1,\, 2,\, \ldots,\, n) \tag{3.19}$$

(no explicit s dependence) where the terms $\tilde{\vartheta}_i\,(y_1,\, y_2,\, \ldots,\, y_n)$ represent analytical functions, can be solved by Lie-series [3]:

$$y_i(s) \;=\; e^{[(s-s_0)D]}\, \hat{y}_i \tag{3.20a}$$

with

$$D \;=\; \tilde{\vartheta}_1\,(\hat{y}_1,\, \hat{y}_2,\, \ldots,\, \hat{y}_n) \cdot \frac{\partial}{\partial \hat{y}_1} + \tilde{\vartheta}_2\,(\hat{y}_1,\, \hat{y}_2,\, \ldots,\, \hat{y}_n) \cdot \frac{\partial}{\partial \hat{y}_2} + \ldots + \tilde{\vartheta}_n\,(\hat{y}_1,\, \hat{y}_2,\, \ldots\, \hat{y}_n) \cdot \frac{\partial}{\partial \hat{y}_n} \tag{3.20b}$$

and

$$y_i(s_0) \;\equiv\; \hat{y}_i \; . \tag{3.20c}$$

Applying this result to the canonical equations of motion:

$$\begin{aligned}
\tilde{\vartheta}_1 &= +\frac{\partial}{\partial p_x}\, \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \\
\tilde{\vartheta}_2 &= -\frac{\partial}{\partial x}\, \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \\
\tilde{\vartheta}_3 &= +\frac{\partial}{\partial p_z}\, \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \\
\tilde{\vartheta}_4 &= -\frac{\partial}{\partial z}\, \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \\
\tilde{\vartheta}_5 &= +\frac{\partial}{\partial p_\sigma}\, \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \\
\tilde{\vartheta}_6 &= -\frac{\partial}{\partial \sigma}\, \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma)
\end{aligned}$$

we obtain:

$$y_1(s) \;\equiv\; x(s) \;=\; e^{(s-s_0)\,D}\, \hat{x} \; ; \tag{3.21a}$$

$$y_2(s) \;\equiv\; p_x(s) \;=\; e^{(s-s_0)\,D}\, \hat{p}_x \; ; \tag{3.21b}$$

$$y_3(s) \;\equiv\; z(s) \;=\; e^{(s-s_0)\,D}\, \hat{z} \; ; \tag{3.21c}$$

$$y_4(s) \;\equiv\; p_z(s) \;=\; e^{(s-s_0)\,D}\, \hat{p}_z \; ; \tag{3.21d}$$

$$y_5(s) \;\equiv\; \sigma(s) \;=\; e^{(s-s_0)\,D}\, \hat{\sigma} \; ; \tag{3.21e}$$

$$y_6(s) \;\equiv\; p_\sigma(s) \;=\; e^{(s-s_0)\,D}\, \hat{p}_\sigma \tag{3.21f}$$

with

$$D = \left[\frac{\partial}{\partial \hat{p}_x} \mathcal{H}(\vec{\hat{y}})\right] \frac{\partial}{\partial \hat{x}} - \left[\frac{\partial}{\partial \hat{x}} \mathcal{H}(\vec{\hat{y}})\right] \frac{\partial}{\partial \hat{p}_x}$$
$$+ \left[\frac{\partial}{\partial \hat{p}_z} \mathcal{H}(\vec{\hat{y}})\right] \frac{\partial}{\partial \hat{z}} - \left[\frac{\partial}{\partial \hat{z}} \mathcal{H}(\vec{\hat{y}})\right] \frac{\partial}{\partial \hat{p}_z}$$
$$+ \left[\frac{\partial}{\partial \hat{p}_\sigma} \mathcal{H}(\vec{\hat{y}})\right] \frac{\partial}{\partial \hat{\sigma}} - \left[\frac{\partial}{\partial \hat{\sigma}} \mathcal{H}(\vec{\hat{y}})\right] \frac{\partial}{\partial \hat{p}_\sigma} \tag{3.22}$$

and

$$\hat{x} \equiv x(s_0) \; ; \quad \hat{p_x} \equiv p_x(s_0) \; ;$$
$$\hat{z} \equiv z(s_0) \; ; \quad \hat{p_z} \equiv p_z(s_0) \; ;$$
$$\hat{\sigma} \equiv \sigma(s_0) \; ; \quad \hat{p_\sigma} \equiv p_\sigma(s_0) \; . \tag{3.23}$$

Remarks:

1) Using the notation of Ref. [4], eqn. (3.20a) may also be written in the form:

$$y_i(s) = e^{:(s-s_0)\mathcal{H}:} \, \hat{y}_i$$

if the autonomous equations of motion result from an Hamiltonian $\mathcal{H}$. So when the approach in Ref. [3] is restricted to canonical systems it is identical to the Lie Algebra method introduced by Dragt.

2) Since the equations of motion (3.6) for a drift space represent an autonomous system of differential equations, eqns. (3.20a, b, c) can be used to determine the transport map $\underline{T}_D$ of a drift space.

In this case we get by comparing (3.19) with (3.6):

$$\tilde{\vartheta}_1 = \frac{y_2}{[1+f(y_6)]} \; ; \tag{3.24a}$$

$$\tilde{\vartheta}_2 = 0 \; ; \tag{3.24b}$$

$$\tilde{\vartheta}_3 = \frac{y_4}{[1+f(y_6)]} \; ; \tag{3.24c}$$

$$\tilde{\vartheta}_4 = 0 \; ; \tag{3.24d}$$

$$\tilde{\vartheta}_5 = 1 - f'(y_6) - \frac{1}{2} \cdot [y_2^2 + y_4^2] \cdot \frac{f'(y_6)}{[1+f(y_6)]^2} \; ; \tag{3.24e}$$

$$\tilde{\vartheta}_6 = 0 \; . \tag{3.24f}$$

This leads to

$$y_i(s_0 + l) = e^{l \cdot D} \, \hat{y}_i \tag{3.25a}$$

with

$$D \;=\; \tilde{\vartheta}_1\left(\hat{\vec{y}}\right)\cdot\frac{\partial}{\partial\hat{y}_1}+\tilde{\vartheta}_3\left(\hat{\vec{y}}\right)\cdot\frac{\partial}{\partial\hat{y}_3}+\tilde{\vartheta}_5\left(\hat{\vec{y}}\right)\cdot\frac{\partial}{\partial\hat{y}_5} \tag{3.25b}$$

and

$$\hat{y}_i \;\equiv\; y_i(s_0)\ . \tag{3.25c}$$

We then have:

$$D\,\hat{y}_1 \;=\; \frac{\hat{y}_2}{[1+f(\hat{y}_6)]}\ ; \tag{3.26a}$$

$$D\,\hat{y}_2 \;=\; 0\ ; \tag{3.26b}$$

$$D\,\hat{y}_3 \;=\; \frac{\hat{y}_4}{[1+f(\hat{y}_6)]}\ ; \tag{3.26c}$$

$$D\,\hat{y}_4 \;=\; 0\ ; \tag{3.26d}$$

$$D\,\hat{y}_5 \;=\; 1-f'(\hat{y}_6)-\frac{1}{2}\cdot[\hat{y}_2^2+\hat{y}_4^2]\cdot\frac{f'(\hat{y}_6)}{[1+f(\hat{y}_6)]^2}\ ; \tag{3.26e}$$

$$D\,\hat{y}_6 \;=\; 0 \tag{3.26f}$$

and

$$D^{\nu}\,\hat{\vec{y}} \;=\; \vec{0}\ \text{ for }\ \nu\ >\ 1\ .$$

Thus:

$$\vec{y}(s_0+l) \;=\; [1+l\cdot D]\ \hat{\vec{y}}\ . \tag{3.27}$$

Putting (3.26) into (3.27), we regain eqn. (3.7).

3) The method for calculating thin-lens transport maps described in this paper works also in the presence of nonsymplectic terms resulting for instance from radiation damping [5]. One must simply include these terms in D before expanding $\exp\left[\hat{D}\right]$.

3.2.2 Calculation of the Thin-Lens Transport Map for the Central Region.

In order to determine the transport map $\underline{T}_L$ for the central region, we investigate the special case:

$$\tilde{\vartheta}_i\,(\vec{y}) \;=\; \delta(s-s_0)\cdot F_i\,(\vec{y})\ ;\ \ \frac{\partial}{\partial s}F_i \;=\; 0\ . \tag{3.28}$$

Replacing the δ-function $\delta(s-s_0)$ in (3.28) by a step function of height $(1/2\epsilon)$ and length (2ϵ), we obtain in this case from eqn. (3.20):

$$\vec{y}(s) = \left\{\exp\left[(s-[s_0-\epsilon])\cdot\frac{1}{2\,\epsilon}\,\hat{D}\right]\right\}\vec{\hat{y}}$$

$$\text{for}\ \ (s_0-\epsilon \le s \le s_0+\epsilon)$$

with

$$\vec{\hat{y}} \equiv \vec{y}(s_0-\epsilon)$$

and

$$\hat{D} = F_1(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_1} + F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2} + \cdots + F_6(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_6}\ . \tag{3.29a}$$

In particular by putting $s = s_0+\epsilon$ we have:

$$\begin{aligned}\vec{y}(s_0+\epsilon) &= \left\{\exp\left[([s_0+\epsilon]-[s_0-\epsilon])\cdot\frac{1}{2\,\epsilon}\,\hat{D}\right]\right\}\vec{\hat{y}} \\ &= \left\{\exp\,[\hat{D}]\right\}\vec{\hat{y}}\ ,\end{aligned}$$

which leads in the limit $\epsilon \longrightarrow 0$ to:

$$\vec{y}(s_0+0) = \left\{\exp\,[\hat{D}]\right\}\vec{\hat{y}}$$

with

$$\vec{\hat{y}} \equiv \vec{y}(s_0-0)\ .$$

Then by choosing the functions $F_i(\vec{y})$ appearing in (3.29a) as:

$$F_i(\vec{y}) = \vartheta_{Li}(\vec{y};s_0)\cdot\Delta s\ , \tag{3.29b}$$

with $\vartheta_{Li}(\vec{y};s_0)$ given by (3.16), one just gets the transport map $\underline{T}_L$ corresponding to eqn. (3.15) in the form:

$$\underline{T}_L = \exp\,[\hat{D}] \tag{3.29c}$$

as may be seen by comparing (3.28) with the r.h.s. of (3.15).

Remark:

The relation (3.29c) for $\underline{T}_L$ can also be derived by solving the differential equation:

$$\frac{d}{ds}\,\vec{y} = \vec{\vartheta}_L(\vec{y};s_0) \equiv \frac{1}{\Delta s}\cdot\vec{F}(\vec{y})\ , \tag{3.30}$$

which does not contain the δ-function $\delta(s-s_0)$ anymore. Writing the solution of (3.30) in the form

$$\vec{y}(s) = \underline{\tilde{T}}(s,\,s_0)\,\vec{y}(s_0)\ , \tag{3.31}$$

one then obtains:

$$\underline{T}_L \equiv \underline{\tilde{T}}(s_0 + \Delta s, s_0) , \tag{3.32}$$

as can be verified by compararing (3.29) with (3.20) and using (3.19).

By (3.30) we see in fact that for the central region II the problem reduces to an autonomous one.

For an example see Remark 2) at the end of Appendix C.

4 Thin - Lens Approximation for Various Kinds of Magnets and for Cavities

In this section the thin - lens transport map corresponding to the central region II (see eqn. (3.13)) is calculated for cavities and for various kinds of magnets.

4.1 Quadrupole

4.1.1 Exponentiation

For a quadrupole we have:

$$g \neq 0$$

and

$$K_x = K_z = N = \lambda = \mu = H = V = 0 .$$

Then we obtain from (3.16) and (3.29b):

$$\begin{aligned}
F_1(\vec{y}) &= 0 ; \\
F_2(\vec{y}) &= -g(s_0) \cdot \Delta s \cdot x ; \\
F_3(\vec{y}) &= 0 ; \\
F_4(\vec{y}) &= +g(s_0) \cdot \Delta s \cdot z; \\
F_5(\vec{y}) &= 0 ; \\
F_6(\vec{y}) &= 0 .
\end{aligned}$$

Thus:

$$\hat{D} = F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} \tag{4.1}$$

and

$$\hat{D}\,\vec{\hat{y}} \;=\; \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ 0 \\ F_4(\vec{\hat{y}}) \\ 0 \\ 0 \end{pmatrix} \;=\; \underline{\hat{A}}\,\vec{\hat{y}} \tag{4.2}$$

with

$$\underline{\hat{A}} \;=\; \Delta s \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.3}$$

The transfer matrix $\underline{M}$ defined by

$$\vec{y}(s_0 + 0) \;=\; \underline{M}\,\vec{y}(s_0 - 0) \;\equiv\; \underline{M}\,\vec{\hat{y}}$$

reads as:

$$\begin{aligned} \underline{M} &= \exp\left[\underline{\hat{A}}\right] \\ &= \underline{1} + \underline{\hat{A}} \end{aligned} \tag{4.4}$$

since

$$\hat{D}\,\underline{\hat{A}} \;=\; \underline{\hat{A}}\,\hat{D} \;\Longrightarrow\; \hat{D}^{\nu}\,\vec{\hat{y}} \;=\; \underline{\hat{A}}^{\nu}\,\vec{\hat{y}} \;\Longrightarrow\; \left\{\exp\left[\hat{D}\right]\right\}\,\vec{\hat{y}} \;=\; \left\{\exp\left[\underline{\hat{A}}\right]\right\}\,\vec{\hat{y}}$$

and

$$\underline{\hat{A}}^{\nu} \;=\; \underline{0} \quad \text{for} \quad \nu \;>\; 1 \;.$$

4.1.2 Thin - Lens Transport Map

From (4.4) we obtain [2] :

$$x^f \;=\; x^i \;;$$

$$p_x^f \;=\; p_x^i - g(s_0)\cdot\Delta s\cdot x^i \;;$$

$$z^f \;=\; z^i \;;$$

[2] See also section A.2.2 in Appendix A, where a superposition of quadrupoles, skew quadrupoles, bending magnets, sextupoles and octupoles is investigated.

$$
\begin{aligned}
p_z^f &= p_z^i + g(s_0) \cdot \Delta s \cdot z^i \ ; \\
\sigma^f &= \sigma^i \ ; \\
p_\sigma^f &= p_\sigma^i
\end{aligned}
$$

with

$$
\begin{aligned}
y^i &\equiv y(s_0 - 0) \ ; \\
y^f &\equiv y(s_0 + 0) \ ;
\end{aligned}
$$

$$
(y = x, p_x, z, p_z, \sigma, p_\sigma) \ .
$$

4.2 Skew Quadrupole

4.2.1 Exponentiation

For a skew quadrupole we have:

$$
N \neq 0
$$

and

$$
K_x = K_z = g = \lambda = \mu = H = V = 0 \ .
$$

Thus we get from (3.16) and (3.29b):

$$
\begin{aligned}
F_1(\vec{y}) &= 0 \ ; \\
F_2(\vec{y}) &= +N(s_0) \cdot \Delta s \cdot z \ ; \\
F_3(\vec{y}) &= 0 \ ; \\
F_4(\vec{y}) &= +N(s_0) \cdot \Delta s \cdot x; \\
F_5(\vec{y}) &= 0 \ ; \\
F_6(\vec{y}) &= 0 \ .
\end{aligned}
$$

Thus:

$$
\hat{D} = F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} \tag{4.5}
$$

and

$$\hat{D}\,\vec{\hat{y}} \;=\; \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ 0 \\ F_4(\vec{\hat{y}}) \\ 0 \\ 0 \end{pmatrix} \;=\; \underline{\hat{A}}\,\vec{\hat{y}} \tag{4.6}$$

with

$$\underline{\hat{A}} \;=\; \Delta s \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \;. \tag{4.7}$$

The transfer matrix reads as:

$$\begin{aligned} \underline{M} &= \exp\left[\underline{\hat{A}}\right] \\ &= \underline{1} + \underline{\hat{A}} \end{aligned} \tag{4.8}$$

since

$$\hat{D}\,\underline{\hat{A}} \;=\; \underline{\hat{A}}\,\hat{D}$$

and

$$\underline{\hat{A}}^{\nu} \;=\; \underline{0} \;\text{ for }\; \nu \;>\; 1 \;.$$

4.2.2 Thin - Lens Transport Map

From (4.8) we obtain:

$$\begin{aligned} x^f &= x^i \;; \\ p_x^f &= p_x^i + N(s_0)\cdot\Delta s\cdot z^i \;; \\ z^f &= z^i \;; \\ p_z^f &= p_z^i + N(s_0)\cdot\Delta s\cdot x^i \;; \\ \sigma^f &= \sigma^i \;; \\ p_\sigma^f &= p_\sigma^i \;. \end{aligned}$$

4.3 Bending Magnet

4.3.1 Exponentiation

For a bending magnet we have:

$$K_x^2 + K_z^2 \;\neq\; 0\,; \quad K_x \cdot K_z \;=\; 0$$

and

$$g \;=\; N \;=\; \lambda \;=\; \mu \;=\; H \;=\; V \;=\; 0\,.$$

From (3.16) and (3.29b) we thus obtain:

$$\begin{aligned}
F_1(\vec{y}) &= 0\,; \\
F_2(\vec{y}) &= -[K_x(s_0)]^2 \cdot \Delta s \cdot x + K_x(s_0) \cdot \Delta s \cdot f(p_\sigma)\,; \\
F_3(\vec{y}) &= 0\,; \\
F_4(\vec{y}) &= -[K_z(s_0)]^2 \cdot \Delta s \cdot z + K_z(s_0) \cdot \Delta s \cdot f(p_\sigma)\,; \\
F_5(\vec{y}) &= -\left[K_x(s_0) \cdot x + K_z(s_0) \cdot z\right] \cdot \Delta s \cdot f'(p_\sigma)\,; \\
F_6(\vec{y}) &= 0\,.
\end{aligned}$$

Thus:

$$\hat{D} \;=\; F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} + F_5(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_5} \tag{4.9}$$

and

$$\hat{D}\,\vec{\hat{y}} \;=\; \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ 0 \\ F_4(\vec{\hat{y}}) \\ F_5(\vec{\hat{y}}) \\ 0 \end{pmatrix} \;=\; \underline{\hat{A}}\,\vec{\hat{y}} \tag{4.10}$$

with

$$\underline{\hat{A}} \;=\; \Delta s \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -K_x^2 & 0 & 0 & 0 & 0 & K_x \cdot f(\hat{p}_\sigma) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -K_z^2 & 0 & 0 & K_z \cdot f(\hat{p}_\sigma) \\ -K_x \cdot f'(\hat{p}_\sigma) & 0 & -K_z \cdot f'(\hat{p}_\sigma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{4.11}$$

The transfer matrix reads as :

$$\begin{aligned} \underline{M} &= \exp\left[\underline{\hat{A}}\right] \\ &= \underline{1} + \underline{\hat{A}} \end{aligned} \tag{4.12}$$

since

$$\hat{D}\,\underline{\hat{A}} = \underline{\hat{A}}\,\hat{D}$$

and

$$\underline{\hat{A}}^{\nu} = \underline{0} \text{ for } \nu > 1 \ .$$

4.3.2 Thin - Lens Transport Map

From (4.12) we obtain :

$$\begin{aligned} x^f &= x^i \ ; \\ p_x^f &= p_x^i - [K_x(s_0)]^2 \cdot \Delta s \cdot x^i + K_x(s_0) \cdot \Delta s \cdot f(p_\sigma^i) \ ; \\ z^f &= z^i \ ; \\ p_z^f &= p_z^i - [K_z(s_0)]^2 \cdot \Delta s \cdot z^i + K_z(s_0) \cdot \Delta s \cdot f(p_\sigma^i) \ ; \\ \sigma^f &= \sigma^i - [K_x \cdot x^i + K_z \cdot z^i] \cdot \Delta s \cdot f'(p_\sigma^i) \ ; \\ p_\sigma^f &= p_\sigma^i \ . \end{aligned}$$

4.4 Sextupole

4.4.1 Exponentiation

For a sextupole we have :

$$\lambda \neq 0$$

and

$$K_x = K_z = g = N = \mu = H = V = 0 \ .$$

From (3.16) and (3.29b) we then get:

$$\begin{aligned} F_1(\vec{y}) &= 0 \ ; \\ F_2(\vec{y}) &= -\frac{1}{2}\,\lambda(s_0)\cdot\Delta s\cdot\left[x^2 - z^2\right] \ ; \\ F_3(\vec{y}) &= 0 \ ; \\ F_4(\vec{y}) &= \lambda(s_0)\cdot\Delta s\cdot x\, z \ ; \\ F_5(\vec{y}) &= 0 \ ; \\ F_6(\vec{y}) &= 0 \ . \end{aligned}$$

Thus:

$$\hat{D} \ = \ F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial \hat{y}_2} + F_4(\vec{\hat{y}})\cdot\frac{\partial}{\partial \hat{y}_4} \tag{4.13}$$

and

$$\hat{D}\,\vec{\hat{y}} \ = \ \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ 0 \\ F_4(\vec{\hat{y}}) \\ 0 \\ 0 \end{pmatrix} \ ; \quad \hat{D}^{\nu}\,\vec{\hat{y}} \ = \ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \ \ \nu \ > \ 1 \tag{4.14}$$

$$\Longrightarrow \ \ \left\{\exp\left[\hat{D}\right]\right\}\vec{\hat{y}} \ = \ \vec{\hat{y}} + \hat{D}\,\vec{\hat{y}} \ . \tag{4.15}$$

4.4.2 Thin - Lens Transport Map

From (4.15) we obtain:

$$\begin{aligned} x^f &= x^i \ ; \\ p_x^f &= p_x^i - \frac{1}{2}\,\lambda(s_0)\cdot\Delta s\cdot[(x^i)^2 - (z^i)^2] \ ; \\ z^f &= z^i \ ; \\ p_z^f &= p_z^i + \lambda(s_0)\cdot\Delta s\cdot x^i\, z^i \ ; \\ \sigma^f &= \sigma^i \ ; \\ p_\sigma^f &= p_\sigma^i \ . \end{aligned}$$

4.5 Octupole

4.5.1 Exponentiation

For an octupole we have:

$$\mu \neq 0$$

and

$$K_x = K_z = g = N = \lambda = H = V = 0 \ .$$

Then we obtain from (3.16) and (3.29b):

$$\begin{aligned}
F_1(\vec{y}) &= 0 \ ; \\
F_2(\vec{y}) &= -\frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\left[x^3 - 3\,xz^2\right] \ ; \\
F_3(\vec{y}) &= 0 \ ; \\
F_4(\vec{y}) &= -\frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\left[z^3 - 3\,x^2 z\right] \ ; \\
F_5(\vec{y}) &= 0 \ ; \\
F_6(\vec{y}) &= 0 \ .
\end{aligned}$$

Thus:

$$\hat{D} = F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2} + F_4(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_4} \tag{4.16}$$

and

$$\hat{D}\,\vec{\hat{y}} = \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ 0 \\ F_4(\vec{\hat{y}}) \\ 0 \\ 0 \end{pmatrix} \ ; \quad \hat{D}^{\nu}\,\vec{\hat{y}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \ \ \nu > 1 \tag{4.17}$$

$$\Longrightarrow \ \left\{\exp\left[\hat{D}\right]\right\}\vec{\hat{y}} = \vec{\hat{y}} + \hat{D}\,\vec{\hat{y}} \ . \tag{4.18}$$

4.5.2 Thin - Lens Transport Map

From (4.18) we obtain:

$$\begin{aligned}
x^f &= x^i ; \\
p_x^f &= p_x^i - \frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\Big[(x^i)^3 - 3\,(x^i)\,(z^i)^2\Big] ; \\
z^f &= z^i ; \\
p_z^f &= p_z^i - \frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\Big[(z^i)^3 - 3\,(x^i)^2\,(z^i)\Big] ; \\
\sigma^f &= \sigma^i ; \\
p_\sigma^f &= p_\sigma^i .
\end{aligned}$$

4.6 Synchrotron - Magnet

4.6.1 Exponentiation

For a synchrotron magnet we have:

$$g \neq 0; \quad K_x^2 + K_z^2 \neq 0 \;\text{ with }\; K_x\cdot K_z = 0 \tag{4.19}$$

and

$$N = \lambda = \mu = H = V = 0 .$$

We thus obtain from (3.16) and (3.29b):

$$\begin{aligned}
F_1(\vec{y}) &= 0 ; \\
F_2(\vec{y}) &= -G_1(s_0)\cdot\Delta s\cdot x + K_x(s_0)\cdot\Delta s\cdot f(p_\sigma) ; \\
F_3(\vec{y}) &= 0 ; \\
F_4(\vec{y}) &= -G_2(s_0)\cdot\Delta s\cdot z + K_z(s_0)\cdot\Delta s\cdot f(p_\sigma) ; \\
F_5(\vec{y}) &= [K_x(s_0)\cdot x + K_z(s_0)\cdot z]\cdot\Delta s\cdot f'(p_\sigma) ; \\
F_6(\vec{y}) &= 0 ;
\end{aligned}$$

$$(\,G_1 = K_x^2 + g;\; G_2 = K_z^2 - g\,) .$$

Thus:

$$\hat{D} = F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} + F_5(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_5} \tag{4.20}$$

and

$$\hat{D}\,\vec{\hat{y}} = \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ 0 \\ F_4(\vec{\hat{y}}) \\ F_5(\vec{\hat{y}}) \\ 0 \end{pmatrix} = \underline{\hat{A}}\,\vec{\hat{y}} \tag{4.21}$$

with

$$\underline{\hat{A}} = \Delta s \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -G_1 & 0 & 0 & 0 & 0 & K_x \cdot f(\hat{p}_\sigma) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -G_2 & 0 & 0 & K_z \cdot f(\hat{p}_\sigma) \\ K_x \cdot f'(\hat{p}_\sigma) & 0 & K_z \cdot f'(\hat{p}_\sigma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.22}$$

The transfer matrix reads as:

$$\begin{aligned} \underline{M} &= \exp\left[\underline{\hat{A}}\right] \\ &= \underline{1} + \underline{\hat{A}} \end{aligned} \tag{4.23}$$

since

$$\hat{D}\,\underline{\hat{A}} = \underline{\hat{A}}\,\hat{D}$$

and

$$\underline{\hat{A}}^{\nu} = \underline{0} \text{ for } \nu > 1 .$$

4.6.2 Thin - Lens Transport Map

From (4.23) we obtain:

$$\begin{aligned} x^f &= x^i ; \\ p_x^f &= p_x^i - G_1(s_0) \cdot \Delta s \cdot x^i + K_x(s_0) \cdot \Delta s \cdot f(p_\sigma^i) ; \\ z^f &= z^i ; \\ p_z^f &= p_z^i - G_2(s_0) \cdot \Delta s \cdot z^i + K_z(s_0) \cdot \Delta s \cdot f(p_\sigma^i) ; \\ \sigma^f &= \sigma^i - [K_x(s_0 \cdot x + K_z(s_0) \cdot z] \cdot \Delta s \cdot f'(p_\sigma^i) ; \\ p_\sigma^f &= p_\sigma^i . \end{aligned}$$

4.7 Solenoid

4.7.1 Exponentiation

For a solenoid we have:

$$H \neq 0$$

and

$$K_x = K_z = g = N = \lambda = \mu = V = 0 \; .$$

Using then (3.16) and (3.29b) we obtain:

$$F_1(\vec{y}) = +\frac{H(s_0) \cdot \Delta s}{[1 + f(p_\sigma)]} \cdot z \; ; \tag{4.24a}$$

$$F_2(\vec{y}) = +\frac{H(s_0) \cdot \Delta s}{[1 + f(p_\sigma)]} \cdot [p_z - H(s_0) \cdot x] \; ; \tag{4.24b}$$

$$F_3(\vec{y}) = -\frac{H(s_0) \cdot \Delta s}{[1 + f(p_\sigma)]} \cdot x \; ; \tag{4.24c}$$

$$F_4(\vec{y}) = -\frac{H(s_0) \cdot \Delta s}{[1 + f(p_\sigma)]} \cdot [p_x + H(s_0) \cdot z] \; ; \tag{4.24d}$$

$$F_5(\vec{y}) = -\frac{H(s_0) \cdot \Delta s}{[1 + f(p_\sigma)]} \cdot \frac{f'(p_\sigma)}{[1 + f(p_\sigma)]} \cdot \left\{ \frac{1}{2} \, H(s_0) \cdot [x^2 + z^2] + [p_x \cdot z - p_z \cdot x] \right\} \; ; \tag{4.24e}$$

$$F_6(\vec{y}) = 0 \; . \tag{4.24f}$$

Thus:

$$\hat{D} = F_1(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_1} + F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_3(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_3} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} + F_5(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_5} \tag{4.25}$$

and

$$\hat{D} \begin{pmatrix} \hat{x} \\ \hat{p}_x \\ \hat{z} \\ \hat{p}_z \end{pmatrix} = \begin{pmatrix} F_1(\vec{\hat{y}}) \\ F_2(\vec{\hat{y}}) \\ F_3(\vec{\hat{y}}) \\ F_4(\vec{\hat{y}}) \end{pmatrix} = \underline{\hat{A}}_0 \begin{pmatrix} \hat{x} \\ \hat{p}_x \\ \hat{z} \\ \hat{p} \end{pmatrix} \; ; \tag{4.26a}$$

$$\hat{D} \, \hat{\sigma} = F_5(\vec{\hat{y}}) \; ; \tag{4.26b}$$

$$\hat{D} \, \hat{p}_\sigma = 0 \implies \left\{ \exp \left[\hat{D} \right] \right\} \hat{p}_\sigma = \hat{p}_\sigma \tag{4.26c}$$

with

$$\underline{\hat{A}}_0 \;=\; \Delta s \cdot \frac{1}{[1+f(\hat{p}_\sigma)]} \cdot \begin{pmatrix} 0 & 0 & +H & 0 \\ -H^2 & 0 & 0 & +H \\ -H & 0 & 0 & 0 \\ 0 & -H & -H^2 & 0 \end{pmatrix} . \tag{4.27}$$

We decompose the matrix $\underline{\hat{A}}_0$ into the components $\underline{\hat{A}}_{01}$ and $\underline{\hat{A}}_{02}$:

$$\underline{\hat{A}}_0 \;=\; \underline{\hat{A}}_{01} + \underline{\hat{A}}_{02} \tag{4.28}$$

with

$$\underline{\hat{A}}_{01} \;=\; \Delta s \cdot \frac{H^2}{[1+f(\hat{p}_\sigma)]} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.29a}$$

and

$$\underline{\hat{A}}_{02} \;=\; \Delta s \cdot \frac{H}{[1+f(\hat{p}_\sigma)]} \cdot \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} . \tag{4.29b}$$

The transfer matrix for

$$\vec{y}_0 \;=\; \begin{pmatrix} x \\ p_x \\ z \\ p \end{pmatrix} \tag{4.30}$$

reads as :

$$\underline{M}_0 \;=\; \exp\left[\underline{\hat{A}}_0\right] \tag{4.31}$$

since

$$\hat{D}\,\underline{\hat{A}}_0 \;=\; \underline{\hat{A}}_0\,\hat{D} \;\Longrightarrow\; \hat{D}^\nu\,\vec{\hat{y}}_0 \;=\; \underline{\hat{A}}^\nu\,\vec{\hat{y}}_0 \;\Longrightarrow\; \left\{\exp\left[\hat{D}\right]\right\}\vec{\hat{y}}_0 \;=\; \left\{\exp\left[\underline{\hat{A}}_0\right]\right\}\vec{\hat{y}}_0 \,.$$

Using the relations :

$$\underline{\hat{A}}_{01} \cdot \underline{\hat{A}}_{02} \;=\; \underline{\hat{A}}_{02} \cdot \underline{\hat{A}}_{01} \;\Longrightarrow\; \exp\left[\underline{\hat{A}}\right] \;=\; \exp\left[\underline{\hat{A}}_{01}\right] \cdot \exp\left[\underline{\hat{A}}_{02}\right]$$

and

$$\left[\underline{\hat{A}}_{01}\right]^\nu \;=\; \underline{0} \;\text{ for }\; \nu > 1 \;\Longrightarrow\; \exp\left[\underline{\hat{A}}_{01}\right] \;=\; \underline{1} + \underline{\hat{A}}_{01}$$

as well as

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^{2n} = (-1)^n \cdot \underline{1} \; ;$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^{2n+1} = (-1)^n \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

we get :

$$\exp\left[\underline{\hat{A}}_{02}\right] =$$

$$\sum_{n,=0}^{\infty} \frac{1}{(2n)!} \cdot (-1)^n \cdot (\Delta\Theta)^{2n} \cdot \underline{1}$$

$$+ \sum_{n,=0}^{\infty} \frac{1}{(2n+1)!} \cdot (-1)^n \cdot (\Delta\Theta)^{2n+1} \cdot \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$= \underline{1} \cdot \cos(\Delta\Theta) + \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \sin(\Delta\Theta)$$

$$= \begin{pmatrix} \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) & 0 \\ 0 & \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) \\ -\sin(\Delta\Theta) & 0 & \cos(\Delta\Theta) & 0 \\ 0 & -\sin(\Delta\Theta) & 0 & \cos(\Delta\Theta) \end{pmatrix}$$

with

$$\Delta\Theta = \frac{H \cdot \Delta s}{[1 + f(\hat{p}_\sigma)]} \; . \tag{4.32}$$

Therefore :

$$\underline{M}_0 = \left[\underline{1} + \underline{\hat{A}}_{01}\right] \cdot \begin{pmatrix} \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) & 0 \\ 0 & \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) \\ -\sin(\Delta\Theta) & 0 & \cos(\Delta\Theta) & 0 \\ 0 & -\sin(\Delta\Theta) & 0 & \cos(\Delta\Theta) \end{pmatrix} \; . \tag{4.33}$$

For the variable σ we get from (4.26b) :

$$\hat{D}^2\,\hat{\sigma} = \hat{D}\,F_5(\vec{\hat{y}})$$

$$= \left\{ F_1(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_1} + F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_3(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_3} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} \right\} F_5(\vec{\hat{y}})$$

$$= \frac{H \cdot \Delta s}{[1+f(\hat{p}_\sigma)]} \left\{ \hat{y}_3 \cdot \frac{\partial}{\partial \hat{y}_1} - [H \cdot \hat{y}_1 - \hat{y}_4] \cdot \frac{\partial}{\partial \hat{y}_2} - \hat{y}_1 \cdot \frac{\partial}{\partial \hat{y}_3} - [H \cdot \hat{y}_3 + \hat{y}_2] \cdot \frac{\partial}{\partial \hat{y}_4} \right\}$$

$$\left\{ \frac{(-H) \cdot f'(\hat{p}_\sigma) \cdot \Delta s}{[1+f(\hat{p}_\sigma)]^2} \left[\frac{1}{2} H \cdot (\hat{y}_1^2 + \hat{y}_3^2) + (\hat{y}_2 \cdot \hat{y}_3 - \hat{y}_4 \cdot \hat{y}_1) \right] \right\}$$

$$= \frac{H \cdot \Delta s}{[1+f(\hat{p}_\sigma)]} \cdot \frac{(-H) \cdot f'(\hat{p}_\sigma) \cdot \Delta s}{[1+f(\hat{p}_\sigma)]^2}$$

$$\times \{ \hat{y}_3 \cdot [H \cdot \hat{y}_1 - \hat{y}_4] - [H \cdot \hat{y}_1 - \hat{y}_4] \cdot \hat{y}_3 - \hat{y}_1 \cdot [H \cdot \hat{y}_3 + \hat{y}_2] + [H \cdot \hat{y}_3 + \hat{y}_2] \cdot \hat{y}_1 \}$$

$$= 0 \; ;$$

$$\Longrightarrow \quad \left\{ \exp \left[\hat{D} \right] \right\} \hat{\sigma} = \hat{\sigma} + F_5(\vec{\hat{y}}) + \frac{1}{2!} \cdot \hat{D} \, F_5(\vec{\hat{y}}) + \cdots$$

$$= \hat{\sigma} + F_5(\vec{\hat{y}}) \; . \tag{4.34}$$

<u>Remarks:</u>

1) We have used a separation of the matrix $\underline{\hat{A}}_0$ into the components $\underline{\hat{A}}_{01}$ and $\underline{\hat{A}}_{02}$ with

$$\underline{\hat{A}}_{01} \cdot \underline{\hat{A}}_{02} = \underline{\hat{A}}_{02} \cdot \underline{\hat{A}}_{01}$$

and

$$\underline{\hat{A}}_{01} \quad \text{nilpotent} \; .$$

Since in addition $\underline{\hat{A}}_{02}$ is diagonalizable, eqn. (4.28) represents an additive Jordan decomposition.

2) The decomposition (4.28) factors $\underline{M}_0$ into a rotation map and a focussing map.

4.7.2 Thin - Lens Transport Map

Equations (4.31), (4.34) and (4.26c) finally lead to:

$$\vec{y}_0^{\,f} = \underline{M}_0 \, \vec{y}_0^{\,i} \; ; \tag{4.35a}$$

$$\sigma^f = \sigma^i - \frac{f'(p_\sigma^i)}{[1+f(p_\sigma^i)]} \cdot \Delta\Theta \times \left\{ \frac{1}{2} H(s_0) \cdot \left[(x^i)^2 + (z^i)^2 \right] + \left[p_x^i \cdot z^i - p_z^i \cdot x^i \right] \right\} \; ; \tag{4.35b}$$

$$p_\sigma^f = p_\sigma^i \tag{4.35c}$$

with

$$\Delta\Theta = \frac{H(s_0) \cdot \Delta s}{[1+f(p_\sigma^i)]} \tag{4.35d}$$

and $\underline{M}_0$ given by (4.33), where the matrix $\hat{\underline{A}}_{01}$ appearing in (4.33) takes the form:

$$\hat{\underline{A}}_{01} = \Delta\Theta \cdot H(s_0) \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.35e}$$

(see eqn. (4.29a)).

In Appendix B the superposition of a solenoid with a quadrupole is investigated.

4.8 Cavity

4.8.1 Exponentiation

For a cavity we have:

$$V \neq 0$$

and

$$K_x = K_z = g = N = \lambda = \mu = H = 0 .$$

Then we obtain from (3.16) and (3.29b):

$$F_1(\vec{y}) = 0 ;$$

$$F_2(\vec{y}) = 0 ;$$

$$F_3(\vec{y}) = 0 ;$$

$$F_4(\vec{y}) = 0 ;$$

$$F_5(\vec{y}) = 0 ;$$

$$F_6(\vec{y}) = \frac{1}{\beta_0^2} \cdot \frac{eV(s_0)}{E_0} \cdot \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \cdot \Delta s .$$

Thus:

$$\hat{D} = F_6(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_6} \tag{4.36}$$

and

$$\hat{D}\,\vec{\hat{y}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_6(\vec{\hat{y}}) \end{pmatrix} ; \quad \hat{D}^{\nu}\,\vec{\hat{y}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for } \nu > 1 \tag{4.37}$$

$$\Longrightarrow \quad \left\{\exp\left[\hat{D}\right]\right\}\vec{y} \;=\; \vec{y} + \hat{D}\,\vec{y}\;. \tag{4.38}$$

4.8.2 Thin - Lens Transport Map

From (4.37) and (4.38) we obtain :

$$\begin{aligned}
x^f &= x^i\;; \\
p_x^f &= p_x^i\;; \\
z^f &= z^i\;; \\
p_z^f &= p_z^i\;; \\
\sigma^f &= \sigma^i\;; \\
p_\sigma^f &= p_\sigma^i + \frac{1}{\beta_0^2}\cdot\frac{eV(s_0)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma^i+\varphi\right]\cdot\Delta s\;.
\end{aligned}$$

5 Summary

As a continuation of Ref. [1], we have shown how to solve the nonlinear canonical equations of motion in the framework of the fully six–dimensional formalism using Lie series and exponentiation. Various kinds of magnets (bending magnets, quadrupoles, synchrotron magnets, skew quadrupoles, sextupoles, octupoles, solenoids) and cavities are treated, taking into account the energy dependence of the focusing strength.

In Appendix A we have introduced an improved Hamiltonian (Appendix A). which is exact outside bending magnets and solenoids.

Since the equations of motion are canonical, the transport maps obtained are automatically symplectic.

The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy, and have been incorporated into the computer program SIXTRACK.

Acknowledgments

We wish to thank D. P. Barber for very stimulating and interesting discussions and for carefully reading the manuscript.

Appendix A: Thin - Lens Approximation with an Improved Hamiltonian

In chapter 4 we have used an approximate Hamiltonian which is obtained by a series expansion of the square root

$$\left\{1-\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{[1+f(p_\sigma)]^2}\right\}^{1/2}$$

up to first order in terms of the quantity

$$\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{[1+f(p_\sigma)]^2}\ .$$

The aim of this Appendix is to repeat the calculations of chapter 4 in the absence of solenoids with an improved Hamiltonian by using the unexpanded square root. The new Hamiltonian is again decomposed into a lens part $\mathcal{H}_L$ and a drift component $\mathcal{H}_D$. Then we apply the thin - lens approximation described in chapter 3 by defining the modified Hamiltonian

$$\mathcal{H}_{mod} \;=\; \mathcal{H}_D+\hat{\mathcal{H}}_L\cdot\Delta s\cdot\delta(s-s_0)$$

(see eqn. (3.12)) and solve the equations of motion resulting from the improved Hamiltonian for the regions I, II, and III of eqn. (3.13). It is shown that the thin - lens maps for the central region II obtained earlier remain valid and that corrections induced by the new Hamiltonian only appear in the solutions of the *drift* spaces (regions I and III).

The improved Hamiltonian introduced in this Appendix is exact outside the bending magnets. Inside a bending magnet we neglect nonlinear crossing terms resulting from the curvature. These terms are investigated in Appendix B.

A.1 The Improved Hamiltonian

In the absence of solenoids the Hamiltonian for orbital motion in storage rings takes the form :

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) \;=\; & p_\sigma-[1+f(p_\sigma)]\cdot[1+K_x\cdot x+K_z\cdot z]\cdot\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2}\\ & +\frac{1}{2}\cdot[1+K_x\cdot x+K_z\cdot z]^2-\frac{1}{2}\cdot g\cdot(z^2-x^2)-N\cdot xz\\ & +\frac{\lambda}{6}\cdot(x^3-3\,xz^2)\\ & +\frac{\mu}{24}\cdot(z^4-6\,x^2\,z^2+x^4)\\ & +\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \qquad \text{(A.1)}\end{aligned}$$

(see eqn. (2.1)).

If we neglect nonlinear crossing terms containing the factors $K_x \cdot x$ and $K_z \cdot z$ we may write:

$$
\begin{aligned}
& -[1+f(p_\sigma)] \cdot [1 + K_x \cdot x + K_z \cdot z] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} \\
\approx\ & -[1+f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} - [1+f(p_\sigma)] \cdot [K_x \cdot x + K_z \cdot z] \ .
\end{aligned}
$$

Then we obtain:

$$
\begin{aligned}
\mathcal{H} \ = \ & p_\sigma - [1+f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} \\
& - [K_x \cdot x + K_z \cdot z] \cdot f(p_\sigma) \\
& + \frac{1}{2} [K_x^2 + g] \cdot x^2 + \frac{1}{2} [K_z^2 - g] \cdot z^2 - N \cdot x\, z \\
& + \frac{\lambda}{6} \cdot (x^3 - 3\, x z^2) + \frac{\mu}{24} \cdot (z^4 - 6\, x^2\, z^2 + x^4) \\
& + \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \qquad \text{(A.2)}
\end{aligned}
$$

(constant terms which have no influence on the motion have been dropped).

In particular we get for the Hamiltonian of a drift space :

$$
\mathcal{H}_D(x, p_x, z, p_z, \sigma, p_\sigma; s) \ = \ p_\sigma - [1+f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma]^2} \right\}^{1/2} \ , \qquad \text{(A.3)}
$$

and the lens component of $\mathcal{H}$:

$$
\mathcal{H}_L \ = \ \mathcal{H} - \mathcal{H}_D
$$

then reads:

$$
\begin{aligned}
\mathcal{H}_L(x, p_x, z, p_z, \sigma, p_\sigma; s) \ = \ & - [K_x \cdot x + K_z \cdot z] \cdot f(p_\sigma) \\
& + \frac{1}{2} [K_x^2 + g] \cdot x^2 + \frac{1}{2} [K_z^2 - g] \cdot z^2 - N \cdot x\, z \\
& + \frac{\lambda}{6} \cdot (x^3 - 3\, x z^2) + \frac{\mu}{24} \cdot (z^4 - 6\, x^2\, z^2 + x^4) \\
& + \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \ . \qquad \text{(A.4)}
\end{aligned}
$$

A.2 Equations of Motion

A.2.1 Drift Space

The Hamiltonian (A.3) for a drift space leads to the canonical equations of motion:

$$\begin{aligned}
\frac{d}{ds}\,x &= +\frac{\partial \mathcal{H}_D}{\partial p_x} \\
&= -[1+f(p_\sigma)]\cdot\frac{1}{2}\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma]^2}\right\}^{-1/2}\cdot\frac{(-2\,p_x)}{[1+f(p_\sigma)]^2} \\
&= +\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2}\cdot\frac{p_x}{[1+f(p_\sigma)]}\;; \qquad \text{(A.5a)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,p_x &= -\frac{\partial \mathcal{H}_D}{\partial x} \\
&= 0 \quad\Longrightarrow\quad p_x \;=\; const\;; \qquad \text{(A.5b)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,z &= +\frac{\partial \mathcal{H}_D}{\partial p_z} \\
&= +\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2}\cdot\frac{p_z}{[1+f(p_\sigma)]}\;; \qquad \text{(A.5c)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,p_z &= -\frac{\partial \mathcal{H}_D}{\partial z} \\
&= 0 \quad\Longrightarrow\quad p_z \;=\; const\;; \qquad \text{(A.5d)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\,\sigma &= +\frac{\partial \mathcal{H}_D}{\partial p_\sigma} \\
&= 1-f'(p_\sigma)\cdot\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2} \\
&\quad -[1+f(p_\sigma)]\cdot\frac{1}{2}\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2}\cdot\frac{2\,[p_x^2+p_z^2]}{[1+f(p_\sigma])^3}\cdot f'(p_\sigma) \\
&= 1-f'(p_\sigma)\cdot\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2} \\
&\quad -\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2}\cdot\frac{[p_x^2+p_z^2]}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma)
\end{aligned}$$

$$= \quad 1 - f'(p_\sigma) \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{-1/2} \quad ; \tag{A.5e}$$

$$\frac{d}{ds} p_\sigma \quad = \quad - \frac{\partial \mathcal{H}_D}{\partial \sigma}$$

$$= \quad 0 \quad \Longrightarrow \quad p_\sigma \; = \; const \; . \tag{A.5f}$$

The solution of eqn. (A.5) for a drift space of length l reads as:

$$x^f \quad = \quad x^i + \left\{ 1 - \frac{[(p_x^i)^2 + (p_z^i)^2]}{[1 + f(p_\sigma^i)]^2} \right\}^{-1/2} \cdot \frac{p_x^i}{[1 + f(p_\sigma^i)]} \cdot l \; ; \tag{A.6a}$$

$$p_x^f \quad = \quad p_x^i \; ; \tag{A.6b}$$

$$z^f \quad = \quad z^i + \left\{ 1 - \frac{[(p_x^i)^2 + (p_z^i)^2]}{[1 + f(p_\sigma^i)]^2} \right\}^{-1/2} \cdot \frac{p_z^i}{[1 + f(p_\sigma^i)]} \cdot l \; ; \tag{A.6c}$$

$$p_z^f \quad = \quad p_z^i \; ; \tag{A.6d}$$

$$\sigma^f \quad = \quad \sigma^i + \left[1 - f'(p_\sigma^i) \cdot \left\{ 1 - \frac{[(p_x^i)^2 + (p_z^i)^2]}{[1 + f(p_\sigma^i)]^2} \right\}^{-1/2} \right] \cdot l \; ; \tag{A.6e}$$

$$p_\sigma^f \quad = \quad p_\sigma^i \; . \tag{A.6f}$$

Remarks: 1) Using (A.5a) and (A.5c), eqn. (A.5e) may also be written in the form:

$$\frac{d}{ds} \sigma \quad = \quad 1 - f'(p_\sigma) \cdot \sqrt{1 + (x')^2 + (z')^2} \; . \tag{A.7f}$$

This result can be obtained directly from the defining equation for σ:

$$\sigma \quad = \quad s - v_0 \cdot t(s) \; ;$$

$$\Longrightarrow \quad \frac{d}{ds} \sigma \quad = \quad 1 - v_0 \cdot \frac{dt}{ds}$$

with

$$dt \quad = \quad \frac{1}{v} \cdot \sqrt{ds^2 + dx^2 + dz^2}$$

and leads to

$$\frac{d}{ds} \sigma \quad = \quad 1 - \frac{v_0}{v} \cdot \sqrt{1 + (x')^2 + (z')^2} \; , \tag{A.8f}$$

which agrees with eqn. (A.6), since

$$f'(p_\sigma) \;=\; \frac{v_0}{v}$$

(see Appendix C in Ref. [1]). Since

$$\sqrt{1+(x')^2+(z')^2} \;\approx\; 1+\frac{1}{2}\,[\,(x')^2+(z')^2\,]\;,$$

one may write :

$$\frac{d}{ds}\,\sigma \;\approx\; 1-f'(p_\sigma)\cdot\left\{1+\frac{1}{2}\,[\,(x')^2+(z')^2\,]\right\}\;.$$

This approximation was used in Ref. [1].

2) As in Ref. [1] one obtains for a drift space :

$$\begin{aligned} x'(s) &= const. \quad\Longrightarrow\quad x(s) = x(s_0)+x'(s_0)\cdot(s-s_0)\;;\\ z'(s) &= const. \quad\Longrightarrow\quad z(s) = z(s_0)+z'(s_0)\cdot(s-s_0)\;, \end{aligned}$$

(see eqns.(A.5a, b) and (A.5c, d)) but the connection betwenn y' and p_y $(y \equiv x,\, z)$ is modified (see eqns.(A.5a, c) and (3.7a, c)).

A.2.2 The Central Part

The equation of motion for the central part (region II in eqn. (3.13)) due to the Hamiltonian (A.2) reads as :

$$\frac{d}{ds}\,\vec{y}(s) \;=\; \vec{F}(\vec{y})\cdot\delta(s-s_0) \qquad \text{(A.9a)}$$

with

$$\vec{F}(\vec{y}) \;=\; \vec{\vartheta}_L(\vec{y};s_0)\cdot\Delta s \qquad \text{(A.9b)}$$

and with $\vec{\vartheta}_L(\vec{y};s)$ given by :

$$\begin{aligned} \vartheta_{L1}(\vec{y};s) &= +\frac{\partial}{\partial p_x}\,\mathcal{H}_L(x,p_x,z,p_z,\sigma,p_\sigma;s)\\ &= 0\;; \qquad \text{(A.10a)} \end{aligned}$$

$$\begin{aligned} \vartheta_{L2}(\vec{y};s) &= -\frac{\partial}{\partial x}\,\mathcal{H}_L(x,p_x,z,p_z,\sigma,p_\sigma;s)\\ &= +K_x(s)\cdot f(p_\sigma)-[K_x^2(s)+g(s)]\cdot x+N(s)\cdot z\\ &\qquad -\frac{\lambda(s)}{2}\cdot(x^2-z^2)-\frac{\mu(s)}{6}\cdot(x^3-3\,x\,z^2)\;; \qquad \text{(A.10b)} \end{aligned}$$

$$\vartheta_{L3}(\vec{y};s) \;=\; +\frac{\partial}{\partial p_z}\,\mathcal{H}_L(x,p_x,z,p_z,\sigma,p_\sigma;s)$$

$$= \quad 0 \; ; \tag{A.10c}$$

$$\begin{aligned}
\vartheta_{L4}(\vec{y};s) &= -\frac{\partial}{\partial z}\,\mathcal{H}_L(x,p_x,z,p_z,\sigma,p_\sigma;s) \\
&= +K_z(s)\cdot f(p_\sigma) - [K_z^2(s) - g(s)]\cdot z + N(s)\cdot x \\
&\qquad +\lambda(s)\cdot xz - \frac{\mu(s)}{6}\cdot(z^3 - 3\,x^2\,z)\;;
\end{aligned} \tag{A.10d}$$

$$\begin{aligned}
\vartheta_{L5}(\vec{y};s) &= +\frac{\partial}{\partial p_\sigma}\,\mathcal{H}_L(x,p_x,z,p_z,\sigma,p_\sigma;s) \\
&= -[K_x(s)\cdot x + K_z(s)\cdot z]\cdot f'(p_\sigma)\;;
\end{aligned} \tag{A.10e}$$

$$\begin{aligned}
\vartheta_{L6}(\vec{y};s) &= -\frac{\partial}{\partial \sigma}\,\mathcal{H}_L(x,p_x,z,p_z,\sigma,p_\sigma;s) \\
&= \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]
\end{aligned} \tag{A.10f}$$

resulting from the Hamiltonian $\mathcal{H}_L$ in eqn. (A.4).

Since the relations (A.10a–f) coincide with eqns. (3.16a–f) in the absence of solenoids ($H = 0$), the thin-lens transport maps calculated in sections 4.1–4.7 remain valid also for the Hamiltonian (A.2). Thus the corrections resulting from the new Hamiltonian are fully absorbed in the solutions for the drift space as can be seen by comparing (A.6) with (3.7).

As an example we consider the superposition of quadrupoles, skew quadrupoles, bending magnets, sextupoles and octupoles and obtain from (A.10):

$$F_1(\vec{y}) = 0\;;$$

$$\begin{aligned}
F_2(\vec{y}) = \Big\{ & K_x(s_0)\cdot f(p_\sigma) - G_1(s_0)\cdot x + N(s_0)\cdot z - \frac{\lambda(s_0)}{2}\cdot(x^2 - z^2) \\
& - \frac{\mu(s_0)}{6}\cdot(x^3 - 3\,x\,z^2)\Big\}\cdot\Delta s\;;
\end{aligned}$$

$$F_3(\vec{y}) = 0\;;$$

$$\begin{aligned}
F_4(\vec{y}) = \{ & K_z(s_0)\cdot f(p_\sigma) - G_2(s_0)\cdot z + N(s_0)\cdot x + \lambda(s_0)\cdot xz \\
& - \frac{\mu(s_0)}{6}\cdot(z^3 - 3\,x^2\,z)\Big\}\cdot\Delta s\;;
\end{aligned}$$

$$F_5(\vec{y}) = [K_x(s_0)\cdot x + K_z(s_0)\cdot z]\cdot f'(p_\sigma)\cdot\Delta s\;;$$

$$F_6(\vec{y}) = 0\;;$$

$$(\,G_1 = K_x^2 + g\,;\; G_2 = K_z^2 - g\,)\;.$$

Thus:

$$\hat{D} \;=\; F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_4} + F_5(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_5} \;. \tag{A.11}$$

We then have:

$$\hat{D}\,\hat{y}_1 \;=\; 0 \;; \tag{A.12a}$$

$$\hat{D}\,\hat{y}_2 \;=\; \left\{ K_x(s_0) \cdot f(p_\sigma) - G_1(s_0) \cdot x + N(s_0) \cdot z - \frac{\lambda(s_0)}{2} \cdot (x^2 - z^2) - \frac{\mu(s_0)}{6} \cdot (x^3 - 3\,x\,z^2) \right\} \cdot \Delta s \;; \tag{A.12b}$$

$$\hat{D}\,\hat{y}_3 \;=\; 0 \;; \tag{A.12c}$$

$$\hat{D}\,\hat{y}_4 \;=\; \left\{ K_z(s_0) \cdot f(p_\sigma) - G_2(s_0) \cdot z + N(s_0) \cdot x + \lambda(s_0) \cdot xz - \frac{\mu(s_0)}{6} \cdot (z^3 - 3\,x^2\,z) \right\} \cdot \Delta s \;; \tag{A.12d}$$

$$\hat{D}\,\hat{y}_5 \;=\; [K_x(s_0) \cdot x + K_z(s_0) \cdot z] \cdot f'(p_\sigma) \cdot \Delta s \;; \tag{A.12e}$$

$$\hat{D}\,\hat{y}_6 \;=\; 0 \tag{A.12f}$$

and

$$\hat{D}^{\nu}\,\vec{\hat{y}} \;=\; \vec{0} \;\text{ for }\; \nu \;>\; 1 \;.$$

Thus:

$$\vec{y}(s_0 + 0) \;=\; \vec{\hat{y}} + \hat{D}\,\vec{\hat{y}} \tag{A.13}$$

with

$$\vec{\hat{y}} \;\equiv\; \vec{y}(s_0)$$

and $\hat{D}\,\vec{\hat{y}}$ given by (A.12).

Equation (A.13) contains as special cases the transport maps of simple quadrupoles, skew quadrupoles, bending magnets, sextupoles and octupoles which are identical with those already derived in section 4.

<u>Remarks:</u>

1) As in chapter 4 the transport maps

$$\vec{y}\left(s_0 - \frac{1}{2}\,\Delta s\right) \;\longrightarrow\; \vec{y}\left(s_0 + \frac{1}{2}\,\Delta s\right)$$

described by a composition of (A.6) and (A.13) (combining the regions I, II and III in eqn. (3.13)) are symplectic for an arbitrary Δs due to the canonical structure of the equations of motion. Furthermore one obtains the exact solution corresponding to the Hamiltonian (A.2) for $\Delta s \longrightarrow 0$.

2) The Hamiltonian (A.2) is exact for a straight section with

$$K_x \; = \; K_z \; = \; 0 \; ,$$

i.e. outside the bending magnets. For a bending magnet with

$$K_x^2 + K_z^2 \; \neq \; 0 \; ; \quad K_x \cdot K_z \; = \; 0$$

the exact Hamiltonian reads as:

$$\begin{aligned}
\mathcal{H}_{bend} \; &= \; p_\sigma - [1 + f(p_\sigma)] \cdot [1 + K_x \cdot x + K_z \cdot z] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} \\
&\quad + \frac{1}{2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 - \frac{1}{2} \\
&= \; \mathcal{H}_D - [1 + f(p_\sigma)] \cdot [K_x \cdot x + K_z \cdot z] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} \\
&\quad + [K_x \cdot x + K_z \cdot z] + \frac{1}{2} K_x^2 \cdot x^2 + \frac{1}{2} K_z^2 \cdot z^2 \\
&= \; \mathcal{H}_D \\
&\quad - [1 + f(p_\sigma)] \cdot [K_x \cdot x + K_z \cdot z] \cdot \left(\left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) \\
&\quad - f(p_\sigma) \cdot [K_x \cdot x + K_z \cdot z] + \frac{1}{2} K_x^2 \cdot x^2 + \frac{1}{2} K_z^2 \cdot z^2
\end{aligned} \tag{A.14}$$

with

$$\mathcal{H}_D \; = \; p_\sigma - [1 + f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma]^2} \right\}^{1/2} \; . \tag{A.15}$$

The drift part $\mathcal{H}_D$ of the bending Hamiltonian coincides with eqn. (A.3). Thus for the regions I) and III) we have again the solution (A.6).

For the central part described by the equations:

$$\vartheta_{L1}(\vec{y}; s_0) \; = \; + \frac{\partial}{\partial p_x} \hat{\mathcal{H}}_{bend}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \tag{A.16a}$$

$$\vartheta_{L2}(\vec{y}; s_0) \; = \; - \frac{\partial}{\partial x} \hat{\mathcal{H}}_{bend}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \tag{A.16b}$$

$$\vartheta_{L3}(\vec{y}; s_0) \; = \; + \frac{\partial}{\partial p_z} \hat{\mathcal{H}}_{bend}(x, p_x, z, p_z, \sigma, p_\sigma) \; ; \tag{A.16c}$$

$$\vartheta_{L4}(\vec{y}; s_0) \;=\; -\frac{\partial}{\partial z}\, \hat{\mathcal{H}}_{bend}(x, p_x, z, p_z, \sigma, p_\sigma)\ ; \tag{A.16d}$$

$$\vartheta_{L5}(\vec{y}; s_0) \;=\; +\frac{\partial}{\partial p_\sigma}\, \hat{\mathcal{H}}_{bend}(x, p_x, z, p_z, \sigma, p_\sigma)\ ; \tag{A.16e}$$

$$\vartheta_{L6}(\vec{y}; s_0) \;=\; -\frac{\partial}{\partial \sigma}\, \hat{\mathcal{H}}_{bend}(x, p_x, z, p_z, \sigma, p_\sigma) \tag{A.16f}$$

(see eqns. (3.10) and (3.11)) we obtain the Hamiltonian:

$$\begin{aligned}\hat{\mathcal{H}}_{bend} \;=\;& -[1+f(p_\sigma)]\cdot[K_x(s_0)\cdot x + K_z(s_0)\cdot z]\cdot\left(\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2}-1\right)\\ & -f(p_\sigma)\cdot[K_x(s_0)\cdot x + K_z(s_0)\cdot z] + \frac{1}{2}\,[K_x(s_0)]^2\cdot x^2 + \frac{1}{2}\,[K_z(s_0)]^2\cdot z^2\ . \end{aligned}\tag{A.17}$$

Using (A.9b) and (3.29), we can calculate $\vec{y}(s_0+0)$ up to an arbitrary order by a series expansion of $\exp[\hat{D}]$. This shall be done in Appendix B where it is shown how to express the transport map by elementary functions (without disturbing the symplecticity), restricting the crossing terms to the lowest order and using a special decomposition of $\hat{\mathcal{H}}_{bend}$ into two parts.

Appendix B: Bending Magnet with Nonlinear Crossing Terms Resulting from the Curvature

B.1 The Hamiltonian

By expanding the Hamiltonian (A.14) and keeping the nonlinear crossing terms of lowest order resulting from the curvature K_x, the whole Hamiltonian of a horizontal bending magnet [3] ($K_x \neq 0$; $K_z = 0$) can be written in the form:

$$\mathcal{H}_{bend} \;=\; \mathcal{H}_D + \mathcal{H}_L \tag{B.1}$$

with a lens component

$$\mathcal{H}_L \;=\; -K_x(s)\cdot x\cdot f(p_\sigma) + \frac{1}{2}\,[K_x(s)]^2\cdot x^2 + \frac{1}{2}\cdot\frac{K_x(s)}{[1+f(p_\sigma)]}\cdot x\cdot\left[p_x^2+p_z^2\right] \tag{B.2}$$

and a drift component $\mathcal{H}_D$ given by (3.5) or more precisely by (A.3).

In order to obtain a thin-lens approximation of a horizontal bending magnet due to the Hamiltonian (B.1) which can be expressed by elementary functions, we decompose the lens component $\mathcal{H}_L$ of eqn. (B.2) into two parts:

$$\mathcal{H}_L \;=\; \mathcal{H}_L^{(1)} + \mathcal{H}_L^{(2)} \tag{B.3}$$

with

$$\mathcal{H}_L^{(1)} \;=\; -K_x(s)\cdot x\cdot f(p_\sigma) + \frac{1}{2}\,[K_x(s)]^2\cdot x^2 + \frac{1}{2}\cdot\frac{K_x(s)}{[1+f(p_\sigma)]}\cdot x\cdot p_z^2\ ; \tag{B.4a}$$

$$\mathcal{H}_L^{(2)} \;=\; \frac{1}{2}\cdot\frac{K_x(s)}{[1+f(p_\sigma)]}\cdot x\cdot p_x^2\ . \tag{B.4b}$$

[3] A vertical bending magnet can be dealt with analogously.

For each part we then separately construct a thin lens (placing one lens behind the other). This is achieved replacing $\mathcal{H}_{bend}$ in (B.1) by

$$\mathcal{H}_{bend} = \mathcal{H}_D + \delta(s - [s_0 - 0]) \cdot \Delta s \cdot \hat{\mathcal{H}}_L^{(1)} + \delta(s - [s_0 + 0]) \cdot \Delta s \cdot \hat{\mathcal{H}}_L^{(2)} \qquad \text{(B.5)}$$

with

$$\hat{\mathcal{H}}_L^{(1)} = -K_x(s_0) \cdot x \cdot f(p_\sigma) + \frac{1}{2}\,[K_x(s_0)]^2 \cdot x^2 + \frac{1}{2} \cdot \frac{K_x(s_0)}{[1 + f(p_\sigma)]} \cdot x \cdot p_z^2 \;; \qquad \text{(B.6a)}$$

$$\hat{\mathcal{H}}_L^{(2)} = \frac{1}{2} \cdot \frac{K_x(s_0)}{[1 + f(p_\sigma)]} \cdot x \cdot p_x^2 \;. \qquad \text{(B.6b)}$$

B.2 Thin - Lens Approximation

B.2.1 The Component $\hat{\mathcal{H}}_L^{(1)}$

From eqns. (3.29b) and (B.6a) we have:

$$\begin{aligned} F_1(\vec{y}) &= +\Delta s \cdot \frac{\partial}{\partial p_x}\, \hat{\mathcal{H}}_L^{(1)}(x, p_x, z, p_z, \sigma, p_\sigma) \\ &= 0 \;; \end{aligned} \qquad \text{(B.7a)}$$

$$\begin{aligned} F_2(\vec{y}) &= -\Delta s \cdot \frac{\partial}{\partial x}\, \hat{\mathcal{H}}_L^{(1)}(x, p_x, z, p_z, \sigma, p_\sigma) \\ &= \Delta s \cdot \left\{ -[K_x(s_0)]^2 \cdot x + K_x(s_0) \cdot f(p_\sigma) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1 + f(p_\sigma)]} \cdot p_z^2 \right\}; \end{aligned} \qquad \text{(B.7b)}$$

$$\begin{aligned} F_3(\vec{y}) &= +\Delta s \cdot \frac{\partial}{\partial p_z}\, \hat{\mathcal{H}}_L^{(1)}(x, p_x, z, p_z, \sigma, p_\sigma) \\ &= \Delta s \cdot \frac{K_x(s_0)}{[1 + f(p_\sigma)]} \cdot x \cdot p_z \;; \end{aligned} \qquad \text{(B.7c)}$$

$$\begin{aligned} F_4(\vec{y}) &= -\Delta s \cdot \frac{\partial}{\partial z}\, \hat{\mathcal{H}}_L^{(1)}(x, p_x, z, p_z, \sigma, p_\sigma) \\ &= 0 \;; \end{aligned} \qquad \text{(B.7d)}$$

$$\begin{aligned} F_5(\vec{y}) &= +\Delta s \cdot \frac{\partial}{\partial p_\sigma} \hat{\mathcal{H}}_L^{(1)}(x, p_x, z, p_z, \sigma, p_\sigma) \\ &= \Delta s \cdot \left\{ -K_x(s_0) \cdot x \cdot f'(p_\sigma) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1 + f(p_\sigma)]} \cdot \frac{f'(p_\sigma)}{[1 + f(p_\sigma)]} \cdot \left[x \cdot p_z^2\right] \right\} \;; \end{aligned} \qquad \text{(B.7e)}$$

$$\begin{aligned} F_6(\vec{y}) &= -\Delta s \cdot \frac{\partial}{\partial \sigma}\, \hat{\mathcal{H}}_L^{(1)}(x, p_x, z, p_z, \sigma, p_\sigma) \\ &= 0 \,. \end{aligned} \qquad \text{(B.7f)}$$

Thus :

$$\hat{D} \;=\; F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} + F_3(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_3} + F_5(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_5} \tag{B.8}$$

and

$$\hat{D}\, \vec{\hat{y}} \;=\; \begin{pmatrix} 0 \\ F_2(\vec{\hat{y}}) \\ F_3(\vec{\hat{y}}) \\ 0 \\ F_5(\vec{\hat{y}}) \\ 0 \end{pmatrix} \;. \tag{B.9}$$

We then get :

a) For x :

$$\hat{D}\, \hat{x} \;=\; 0$$

$$\Longrightarrow \;\; \hat{D}^{\nu}\, \hat{x} \;=\; 0 \;\text{ for }\; \nu \;>\; 0 \;.$$

Thus :

$$\left\{\exp\,[\hat{D}]\right\} \hat{x} \;\equiv\; \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \hat{D}^n\, \hat{x} \;=\; \hat{x} \;. \tag{B.10}$$

b) For p_x :

$$\begin{aligned}
\hat{D}\, \hat{p}_x \;&=\; F_2(\vec{\hat{y}}) \\
&=\; \Delta s \cdot \left\{ -[K_x(s_0)]^2 \cdot \hat{x} + K_x(s_0) \cdot f(\hat{p}_\sigma) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{p}_\sigma)]} \cdot \hat{p}_z^2 \right\} \\
&=\; \Delta s \cdot \left\{ -[K_x(s_0)]^2 \cdot \hat{y}_1 + K_x(s_0) \cdot f(\hat{y}_6) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{y}_6)]} \cdot \hat{y}_4^2 \right\} \;; \\
\hat{D}^2\, \hat{p}_x \;&=\; \Delta s \cdot \hat{D} \left\{ -[K_x(s_0)]^2 \cdot \hat{y}_1 + K_x(s_0) \cdot f(\hat{y}_6) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{y}_6)]} \cdot \hat{y}_4^2 \right\} \\
&=\; 0 \;;
\end{aligned}$$

$$\Longrightarrow \;\; \hat{D}^{\nu}\, \hat{p}_x \;=\; 0 \;\text{ for }\; \nu \;>\; 1 \;.$$

Thus :

$$\begin{aligned}
\left\{\exp\,[\hat{D}]\right\} \hat{p}_x \;&=\; \hat{p}_x + \hat{D}\, \hat{p}_x \\
&=\; \hat{y}_2 + \Delta s \cdot \left\{ -[K_x(s_0)]^2 \cdot \hat{y}_1 + K_x(s_0) \cdot f(\hat{y}_6) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{y}_6)]} \cdot \hat{y}_4^2 \right\} \\
&\equiv\; \hat{p}_x + \Delta s \cdot \left\{ -[K_x(s_0)]^2 \cdot \hat{x} + K_x(s_0) \cdot f(\hat{p}_\sigma) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{p}_\sigma)]} \cdot \hat{p}_z^2 \right\} \;;
\end{aligned} \tag{B.11}$$

c) For z:

$$\hat{D}\,\hat{z} \;=\; F_3(\vec{\hat{y}}) \;=\; \Delta s \cdot \frac{K_x(s_0)}{[1+f(p_\sigma)]} \cdot [\hat{y}_1\,\hat{y}_4] \;;$$

$$\hat{D}^2\,\hat{z} \;=\; \Delta s \cdot \frac{K_x(s_0)}{[1+f(p_\sigma)]} \cdot \hat{D}\,[\hat{y}_1\,\hat{y}_4] \;=\; 0 \;;$$

$$\Longrightarrow \quad \hat{D}^\nu\,\hat{z} \;=\; 0 \;\text{ for }\; \nu \;>\; 1 \;.$$

Thus:

$$\begin{aligned}\left\{\exp\,[\hat{D}]\right\}\hat{z} \;&=\; \hat{y}_3 + \Delta s \cdot \frac{K_x(s_0)}{[1+f(p_\sigma)]} \cdot [\hat{y}_1\,\hat{y}_4] \\ &\equiv\; \hat{z} + \Delta s \cdot \frac{K_x(s_0)}{[1+f(p_\sigma)]} \cdot [\hat{x}\,\hat{p}_z] \;. \end{aligned} \tag{B.12}$$

d) For p_z:

$$\hat{D}\,\hat{p}_z \;=\; 0 \;;$$

$$\Longrightarrow \quad \hat{D}^\nu\,\hat{p}_z \;=\; 0 \;\text{ for }\; \nu \;>\; 0 \;.$$

Thus:

$$\left\{\exp\,[\hat{D}]\right\}\hat{p}_z \;=\; \hat{p}_z \;. \tag{B.13}$$

e) For σ:

$$\begin{aligned}\hat{D}\,\hat{\sigma} \;&=\; F_5(\vec{\hat{y}}) \\ &=\; \Delta s \cdot \left\{ -K_x(s_0)\cdot\hat{x}\cdot f'(\hat{p}_\sigma) - \frac{1}{2}\cdot\frac{K_x(s_0)}{[1+f(\hat{p}_\sigma)]}\cdot\frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]}\cdot\left[\hat{x}\,\hat{p}_z^2\right] \right\} \\ &=\; \Delta s \cdot \left\{ -K_x(s_0)\cdot\hat{y}_1\cdot f'(\hat{y}_6) - \frac{1}{2}\cdot\frac{K_x(s_0)}{[1+f(\hat{y}_6)]}\cdot\frac{f'(\hat{y}_6)}{[1+f(\hat{y}_6)]}\cdot\left[\hat{y}_1\,\hat{y}_4^2\right] \right\} \;; \\ \hat{D}^2\,\hat{\sigma} \;&=\; \Delta s \cdot \hat{D}\left\{ -K_x(s_0)\cdot\hat{y}_1\cdot f'(\hat{y}_6) - \frac{1}{2}\cdot\frac{K_x(s_0)}{[1+f(\hat{y}_6)]}\cdot\frac{f'(\hat{y}_6)}{[1+f(\hat{y}_6)]}\cdot\left[\hat{y}_1\,\hat{y}_4^2\right] \right\} \\ &=\; 0 \;; \end{aligned}$$

$$\Longrightarrow \quad \hat{D}^\nu\,\hat{\sigma} \;=\; 0 \;\text{ for }\; \nu \;>\; 1 \;.$$

Thus:

$$\left\{\exp\,[\hat{D}]\right\}\hat{\sigma} \;=\; \hat{\sigma} + \hat{D}\,\hat{\sigma}$$

$$= \hat{y}_5 + \Delta s \cdot \left\{ -K_x(s_0) \cdot \hat{y}_1 \cdot f'(\hat{y}_6) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{y}_6)]} \cdot \frac{f'(\hat{y}_6)}{[1+f(\hat{y}_6)]} \cdot \left[\hat{y}_1 \, \hat{y}_4^2 \right] \right\}$$

$$= \hat{y}_5 + \Delta s \cdot \hat{y}_1 \cdot \left\{ -K_x(s_0) \cdot f'(\hat{y}_6) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{y}_6)]} \cdot \frac{f'(\hat{y}_6)}{[1+f(\hat{y}_6)]} \cdot \hat{y}_4^2 \right\}$$

$$= \hat{\sigma} + \Delta s \cdot \hat{x} \cdot \left\{ -K_x(s_0) \cdot f'(\hat{p}_\sigma) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(\hat{p}_\sigma)]} \cdot \frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]} \cdot \hat{p}_z^2 \right\} . \tag{B.14}$$

f) For p_σ :

$$\hat{D}\, \hat{p}_\sigma = 0 ;$$

$$\Longrightarrow \quad \hat{D}^\nu\, \hat{p}_\sigma = 0 \text{ for } \nu > 0 .$$

Thus:

$$\left\{ \exp\left[\hat{D}\right] \right\} \hat{p}_\sigma = \hat{p}_\sigma . \tag{B.15}$$

Equations (B.10 – 15) finally lead to:

$$x^f = x^i ; \tag{B.16a}$$

$$p_x^f = p_x^i + \Delta s \cdot \left\{ -[K_x(s_0)]^2 \cdot x^i + K_x(s_0) \cdot f(p_\sigma^i) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(p_\sigma^i)]} \cdot (p_z^i)^2 \right\} ; \tag{B.16b}$$

$$z^f = z^i + \Delta s \cdot \frac{K_x(s_0)}{[1+f(p_\sigma^i)]} \cdot [x^i \, p_z^i] ; \tag{B.16c}$$

$$p_z^f = p_z^i ; \tag{B.16d}$$

$$\sigma^f = \sigma^i + \Delta s \cdot x^i \cdot \left\{ -K_x(s_0) \cdot f'(p_\sigma^i) - \frac{1}{2} \cdot \frac{K_x(s_0)}{[1+f(p_\sigma^i)]} \cdot \frac{f'(p_\sigma^i)}{[1+f(p_\sigma^i)]} \cdot (p_z^i)^2 \right\} ; \tag{B.16e}$$

$$p_\sigma^f = p_\sigma^i . \tag{B.16f}$$

B.2.2 The Component $\hat{\mathcal{H}}_L^{(2)}$

From eqns. (3.29b) and (B.6b) we have:

$$F_1(\vec{y}) = +\Delta s \cdot \frac{\partial}{\partial p_x} \hat{\mathcal{H}}_L^{(2)}(x, p_x, z, p_z, \sigma, p_\sigma)$$

$$
\begin{aligned}
&= \frac{K_x(s_0)\cdot\Delta s}{[1+f(p_\sigma)]}\cdot x\cdot p_x \\
&= A\cdot[y_1\,y_2]\ ;
\end{aligned}
$$

$$
\begin{aligned}
F_2(\vec{y}) &= -\Delta s\cdot\frac{\partial}{\partial x}\,\hat{\mathcal{H}}_L^{(2)}(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= -\frac{1}{2}\,\frac{K_x(s_0)\cdot\Delta s}{[1+f(p_\sigma)]}\cdot p_x^2 \\
&= -\frac{1}{2}\,A\cdot y_2^2\ ;
\end{aligned}
$$

$$
\begin{aligned}
F_3(\vec{y}) &= +\Delta s\cdot\frac{\partial}{\partial p_z}\,\hat{\mathcal{H}}_L^{(2)}(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= 0\ ;
\end{aligned}
$$

$$
\begin{aligned}
F_4(\vec{y}) &= -\Delta s\cdot\frac{\partial}{\partial z}\,\hat{\mathcal{H}}_L^{(2)}(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= 0\ ;
\end{aligned}
$$

$$
\begin{aligned}
F_5(\vec{y}) &= +\Delta s\cdot\frac{\partial}{\partial p_\sigma}\,\hat{\mathcal{H}}_L^{(2)}(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= -\frac{1}{2}\,\frac{K_x(s_0)\cdot\Delta s}{[1+f(p_\sigma)]}\cdot\frac{f'(p_\sigma)}{[1+f(p_\sigma)]}\cdot x\cdot p_x^2\ ; \\
&= -\frac{1}{2}\,A\cdot\frac{f'(p_\sigma)}{[1+f(p_\sigma)]}\cdot[y_1\,y_2^2]\ ;
\end{aligned}
$$

$$
\begin{aligned}
F_6(\vec{y}) &= -\Delta s\cdot\frac{\partial}{\partial \sigma}\,\hat{\mathcal{H}}_L^{(2)}(x,p_x,z,p_z,\sigma,p_\sigma) \\
&= 0
\end{aligned}
$$

whereby we have written for abbreviation:

$$
A = \frac{K_x(s_0)\cdot\Delta s}{[1+f(\hat{p}_\sigma)]}\ . \tag{B.17}
$$

Thus:

$$
\begin{aligned}
\hat{D} &= F_1(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_1}+F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2}+F_5(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_5} \\
&= A\cdot\left\{[\hat{y}_1\,\hat{y}_2]\cdot\frac{\partial}{\partial\hat{y}_1}-\frac{1}{2}\,\hat{y}_2^2\cdot\frac{\partial}{\partial\hat{y}_2}-\frac{1}{2}\,\frac{f'(p_\sigma)}{[1+f(p_\sigma)]}\cdot\left[\hat{y}_1\,\hat{y}_2^2\right]\hat{y}_2^2\cdot\frac{\partial}{\partial\hat{y}_5}\right\}
\end{aligned} \tag{B.18}
$$

and

$$
\hat{D}\,\vec{\hat{y}} = \begin{pmatrix} F_1(\vec{\hat{y}}) \\ F_2(\vec{\hat{y}}) \\ 0 \\ 0 \\ F_5(\vec{\hat{y}}) \\ 0 \end{pmatrix}\ . \tag{B.19}
$$

We then obtain:

a) For x:

$$\begin{aligned}
\hat{D}\,\hat{x} &= F_1(\vec{\hat{y}}) = A\cdot[\hat{y}_1\,\hat{y}_2] \;; \\
\hat{D}^2\,\hat{x} &= \hat{D}\,F_1(\vec{\hat{y}}) \\
&= \left\{F_1(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_1} + F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2}\right\} F_1(\vec{\hat{y}}) \\
&= A^2\cdot\left\{[\hat{y}_1\,\hat{y}_2]\cdot\frac{\partial}{\partial\hat{y}_1} - \frac{1}{2}\,\hat{y}_2^2\cdot\frac{\partial}{\partial\hat{y}_2}\right\}[\hat{y}_1\,\hat{y}_2] \\
&= A^2\cdot\left\{\hat{y}_1\cdot\hat{y}_2^2 - \frac{1}{2}\,\hat{y}_2^2\cdot\hat{y}_1\right\} \\
&= A^2\cdot\frac{1}{2}\,\left\{\hat{y}_1\cdot\hat{y}_2^2\right\} \\
&= A^2\cdot\frac{1}{2}\,\hat{y}_1\,\hat{y}_2^2 \;; \\
\hat{D}^3\,\hat{x} &= A^2\cdot\left\{F_1(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_1} + F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2}\right\}\left\{\frac{1}{2}\,\hat{y}_1\,\hat{y}_2^2\right\} \\
&= A^3\cdot\left\{[\hat{y}_1\,\hat{y}_2]\cdot\frac{\partial}{\partial\hat{y}_1} - \frac{1}{2}\,\hat{y}_2^2\cdot\frac{\partial}{\partial\hat{y}_2}\right\}\left\{\frac{1}{2}\,\hat{y}_1\,\hat{y}_2^2\right\} \\
&= A^3\cdot\frac{1}{2}\,\left\{[\hat{y}_1\,\hat{y}_2]\,\hat{y}_2^2 - \frac{1}{2}\,\hat{y}_2^2\cdot 2\;[\hat{y}_1\,\hat{y}_2]\right\} \\
&= 0 \;;
\end{aligned}$$

$$\Longrightarrow\quad \hat{D}^\nu\,\hat{x} = 0 \;\text{ for }\; \nu > 2 \;.$$

Thus:

$$\begin{aligned}
\left\{\exp[\hat{D}]\right\}\hat{x} &\equiv \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\hat{D}^n\,\hat{x} \\
&= \hat{x} + \hat{D}\,\hat{x} + \frac{1}{2}\,\hat{D}^2\,\hat{x} \\
&= \hat{y}_1 + A\cdot[\hat{y}_1\,\hat{y}_2] + \frac{1}{2}\cdot\frac{A^2}{2}\,\hat{y}_1\,\hat{y}_2^2 \\
&= \hat{y}_1\cdot\left\{1 + A\cdot\hat{y}_2 + \frac{1}{4}\,A^2\cdot\hat{y}_2^2\right\} \\
&\equiv \hat{x}\cdot\left\{1 + A\cdot\hat{p}_x + \frac{1}{4}\,A^2\cdot\hat{p}_x^2\right\} \;. \qquad \text{(B.20)}
\end{aligned}$$

b) For p_x:

$$\hat{D}\,\hat{p}_x = F_2(\vec{\hat{y}}) = -\frac{A}{2}\cdot\hat{y}_2^2 \;;$$

$$
\begin{aligned}
\hat{D}^2\,\hat{p}_x &= \hat{D}\,F_2(\vec{\hat{y}}) \\
&= \left\{F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2}\right\}\,F_2(\vec{\hat{y}}) \\
&= \left(-\frac{A}{2}\right)^2\left\{\hat{y}_2^2\cdot\frac{\partial}{\partial\hat{y}_2}\right\}\hat{y}_2^2 \\
&= \left(-\frac{A}{2}\right)^2\cdot\left\{\hat{y}_2^2\cdot 2\,\hat{y}_2\right\} \\
&= \left(-\frac{A}{2}\right)^2\cdot 2\,\hat{y}_2^3
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}^3\,\hat{p}_x &= \left(-\frac{A}{2}\right)^2\cdot 2\cdot\left\{F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2}\right\}\hat{y}_2^3 \\
&= \left(-\frac{A}{2}\right)^3\cdot 2\cdot\left\{\hat{y}_2^2\cdot\frac{\partial}{\partial\hat{y}_2}\right\}\hat{y}_2^3 \\
&= \left(-\frac{A}{2}\right)^3\cdot 2\cdot 3\cdot\hat{y}_2^4\ ;
\end{aligned}
$$

Thus:

$$
\hat{D}^n\,\hat{x} = \left(-\frac{A}{2}\right)^n\cdot(n!)\cdot\hat{y}_2^{n+1}\ ;
$$

$$
\begin{aligned}
\Longrightarrow\quad \left\{\exp\,[\hat{D}]\right\}\hat{p}_x &= \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\hat{D}^n\,\hat{x} \\
&= \sum_{n=0}^{\infty}\left(-\frac{A}{2}\right)^n\cdot\hat{y}_2^{n+1} \\
&= \frac{\hat{y}_2}{1+\frac{A}{2}\,\hat{y}_2} \\
&\equiv \frac{\hat{p}_x}{1+\frac{A}{2}\,\hat{p}_x}\ .
\end{aligned}
\tag{B.21}
$$

c) For z:

$$
\hat{D}\,\hat{z} = 0\ ;
$$

$$
\Longrightarrow\quad \hat{D}^\nu\,\hat{z} = 0\ \text{ for }\ \nu > 0\ .
$$

Thus:

$$
\left\{\exp\,[\hat{D}]\right\}\hat{z} = \hat{z}\ . \tag{B.22}
$$

d) For p_z:

$$
\hat{D}\,\hat{p}_z = 0\ ;
$$

$$\Longrightarrow \quad \hat{D}^{\nu}\, \hat{p}_z \;=\; 0 \;\; \text{for} \;\; \nu \;>\; 0 \; .$$

Thus:

$$\Big\{ \exp\,[\hat{D}] \Big\}\, \hat{p}_z \;=\; \hat{p}_z \; . \tag{B.23}$$

e) For σ:

$$\begin{aligned}
\hat{D}\,\hat{\sigma} &= F_5(\vec{\hat{y}}) \;=\; -\frac{1}{2}\, A \cdot \frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]} \cdot [\hat{y}_1\, \hat{y}_2^{\,2}] \; ; \\
\hat{D}^2\,\hat{\sigma} &= \hat{D}\, F_5(\vec{\hat{y}}) \\
&= \left\{ F_1(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_1} + F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial \hat{y}_2} \right\}\, F_5(\vec{\hat{y}}) \\
&= -\frac{1}{2}\, A^2 \cdot \frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]} \cdot \left\{ [\hat{y}_1\, \hat{y}_2] \cdot \frac{\partial}{\partial \hat{y}_1} - \frac{1}{2}\, \hat{y}_2^2 \cdot \frac{\partial}{\partial \hat{y}_2} \right\} \left[\hat{y}_1\, \hat{y}_2^{2}\right] \\
&= -\frac{1}{2}\, A^2 \cdot \frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]} \cdot \left\{ [\hat{y}_1\, \hat{y}_2] \cdot \hat{y}_2^2 - \frac{1}{2}\, \hat{y}_2^2 \cdot 2\; [\hat{y}_1\, \hat{y}_2] \right\} \\
&= 0 \; ;
\end{aligned}$$

$$\Longrightarrow \quad \hat{D}^{\nu}\, \hat{\sigma} \;=\; 0 \;\; \text{for} \;\; \nu \;>\; 1 \; .$$

Thus:

$$\begin{aligned}
\Big\{ \exp\,[\hat{D}] \Big\}\, \hat{\sigma} &\equiv \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \hat{D}^n\, \hat{\sigma} \\
&= \hat{\sigma} + \hat{D}\, \hat{\sigma} \\
&= \hat{y}_5 - \frac{1}{2}\, A \cdot \frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]} \cdot [\hat{y}_1\, \hat{y}_2^{\,2}] \\
&\equiv \hat{\sigma} - \frac{1}{2}\, A \cdot \frac{f'(\hat{p}_\sigma)}{[1+f(\hat{p}_\sigma)]} \cdot \hat{x}\, \hat{p}_x^{\,2} \; .
\end{aligned} \tag{B.24}$$

f) For p_σ:

$$\hat{D}\, \hat{p}_\sigma \;=\; 0 \; ;$$

$$\Longrightarrow \quad \hat{D}^{\nu}\, \hat{p}_\sigma \;=\; 0 \;\; \text{for} \;\; \nu \;>\; 0 \; .$$

Thus:

$$\Big\{ \exp\,[\hat{D}] \Big\}\, \hat{p}_\sigma \;=\; \hat{p}_\sigma \; . \tag{B.25}$$

From eqns. (B.20 – 25) we finally get:

$$
\begin{aligned}
x^f &= x^i \cdot \left\{ 1 + A \cdot p_x^i + \frac{1}{4} A^2 \cdot (p_x^i)^2 \right\} \\
&= x^i \cdot \left\{ 1 + \frac{1}{2} A \cdot p_x^i \right\}^2 \quad ; \qquad \text{(B.26a)}
\end{aligned}
$$

$$
p_x^f = \frac{p_x^i}{1 + \frac{A}{2} p_x^i} \; ; \qquad \text{(B.26b)}
$$

$$
z^f = z^i \; ; \qquad \text{(B.26c)}
$$

$$
p_z^f = p_z^i \; ; \qquad \text{(B.26d)}
$$

$$
\sigma^f = \sigma^i - \frac{1}{2} A \cdot \frac{f'(p_\sigma^i)}{[1 + f(p_\sigma^i)]} \cdot x^i \cdot (p_x^i)^2 \; ; \qquad \text{(B.26e)}
$$

$$
p_\sigma^f = p_\sigma^i \qquad \text{(B.26f)}
$$

with A given by (B.17).

Remarks:

1) The thin-lens transport map resulting from eqn. (B.5) reads as:

$$
T_L = \exp\left[\hat{D}^{(1)}\right] \cdot \exp\left[\hat{D}^{(2)}\right]
$$

whereby $\hat{D}^{(1)}$ corresponds to the component $\hat{\mathcal{H}}_L^{(1)}$ and $\hat{D}^{(2)}$ to $\hat{\mathcal{H}}_L^{(2)}$. One could also use the more symmetric composition:

$$
T_L = \exp\left[\frac{1}{2} \hat{D}^{(1)}\right] \cdot \exp\left[\hat{D}^{(2)}\right] \cdot \exp\left[\frac{1}{2} \hat{D}^{(1)}\right] .
$$

2) As mentioned in the remark at the end of section 3.2.2, one can calculate the transport map $\underline{T_L}$ alternatively by solving the differential equation (3.30). To illustrate this method we use the lens corresponding to $\hat{\mathcal{H}}_L^{(2)}$. In this case the differential equations (3.30) take the form:

$$
x' = \frac{A}{\Delta s} \cdot x \cdot p_x \; ; \qquad \text{(B.27a)}
$$

$$
p_x{}' = -\frac{1}{2} \cdot \frac{A}{\Delta s} \cdot p_x^2 \; ; \qquad \text{(B.27b)}
$$

$$
z' = 0 \; ; \qquad \text{(B.27c)}
$$

$$
p_z{}' = 0 \; ; \qquad \text{(B.27d)}
$$

$$\sigma' = -\frac{1}{2} \cdot \frac{A}{\Delta s} \cdot \frac{f'(p_\sigma)}{[1+f(p_\sigma)]} \cdot x \cdot p_x^2 \ ; \tag{B.27e}$$

$$p_\sigma{}' = 0 \ , \tag{B.27f}$$

which are solved by:

$$x(s) = x(s_0) \cdot \left[1 + \frac{1}{2} \cdot \frac{A}{\Delta s} \cdot p_x(s_0) \cdot (s - s_0) \right]^2 \ ; \tag{B.28}$$

$$p_x(s) = \frac{p_x(s_0)}{1 + \frac{1}{2} \cdot \frac{A}{\Delta s} \cdot p_x(s_0) \cdot (s - s_0)} \ ; \tag{B.28a}$$

$$z(s) = z(s_0) \ ; \tag{B.28b}$$

$$p_z(s) = p_z(s_0) \ ; \tag{B.28c}$$

$$\sigma(s) = \sigma(s_0) - \frac{1}{2} \cdot \frac{A}{\Delta s} \cdot \frac{f'[p_\sigma(s_0)]}{1+f[p_\sigma(s_0)]} \cdot x(s_0) \cdot p_x^2(s_0) \cdot (s - s_0) \ ; \tag{B.28d}$$

$$p_\sigma(s) = p_\sigma(s_0) \ . \tag{B.28e}$$

Using then the relations:

$$\vec{y}^{\,i} \equiv \vec{y}(s_0) \ ;$$
$$\vec{y}^{\,f} \equiv \vec{y}(s_0 + \Delta s) \ ,$$

(see (3.31) and (3.32)) one indeed regains eqns. (B.26a – f).

Appendix C: Superposition of Solenoids and Quadrupoles

C.1 Exponentiation

For a superposition of a solenoid and a quadrupole we have:

$$H \neq 0; \ g \neq 0$$

and

$$K_x = K_z = N = \lambda = \mu = V = 0 \ .$$

From (3.16) and (3.29b) we then obtain:

$$F_1(\vec{y}) = +\frac{H(s_0) \cdot \Delta s}{[1+f(p_\sigma)]} \cdot z \ ; \tag{C.1}$$

$$F_2(\vec{y}) = +\frac{H(s_0) \cdot \Delta s}{[1+f(p_\sigma)]} \cdot [p_z - H(s_0) \cdot x] - g \cdot x \cdot \Delta s \ ; \tag{C.1a}$$

$$F_3(\vec{y}) \;=\; -\frac{H(s_0)\cdot\Delta s}{[1+f(p_\sigma)]}\cdot x \;; \tag{C.1b}$$

$$F_4(\vec{y}) \;=\; -\frac{H(s_0)\cdot\Delta s}{[1+f(p_\sigma)]}\cdot[p_x + H(s_0)\cdot z] + g\cdot z\cdot\Delta s \;; \tag{C.1c}$$

$$F_5(\vec{y}) \;=\; -\frac{H(s_0)\cdot\Delta s}{[1+f(p_\sigma)]}\cdot\frac{f'(p_\sigma)}{[1+f(p_\sigma)]}\cdot\left\{\frac{1}{2}\,H(s_0)\cdot[x^2+z^2] + [p_x\cdot z - p_z\cdot x]\right\} \;; \tag{C.1d}$$

$$F_6(\vec{y}) \;=\; 0\;. \tag{C.1e}$$

Thus:

$$\hat{D} \;=\; F_1(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_1} + F_2(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_2} + F_3(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_3} + F_4(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_4} + F_5(\vec{\hat{y}})\cdot\frac{\partial}{\partial\hat{y}_5} \tag{C.2}$$

and

$$\hat{D}\begin{pmatrix}\hat{x}\\ \hat{p}_x\\ \hat{z}\\ \hat{p}_z\end{pmatrix} \;=\; \begin{pmatrix}F_1(\vec{\hat{y}})\\ F_2(\vec{\hat{y}})\\ F_3(\vec{\hat{y}})\\ F_4(\vec{\hat{y}})\end{pmatrix} \;=\; \underline{\hat{A}}_0\begin{pmatrix}\hat{x}\\ \hat{p}_x\\ \hat{z}\\ \hat{p}\end{pmatrix} \;; \tag{C.3a}$$

$$\hat{D}\,\hat{\sigma} \;=\; F_5(\vec{\hat{y}}) \;; \tag{C.3b}$$

$$\hat{D}\,\hat{p}_\sigma \;=\; 0 \;\Longrightarrow\; \left\{\exp\left[\hat{D}\right]\right\}\hat{p}_\sigma \;=\; \hat{p}_\sigma \tag{C.3c}$$

with

$$\underline{\hat{A}}_0 \;=\; \Delta s\cdot\frac{1}{[1+f(\hat{p}_\sigma)]}\cdot\begin{pmatrix} 0 & 0 & +H & 0\\ -[H^2+\hat{g}\,] & 0 & 0 & +H\\ -H & 0 & 0 & 0\\ 0 & -H & -[H^2-\hat{g}\,] & 0\end{pmatrix} \tag{C.4}$$

and

$$\frac{\hat{g}}{[1+f(p_\sigma)]} \;=\; g\;. \tag{C.5}$$

We decompose the matrix $\underline{\hat{A}}_0$ into the components $\underline{\hat{A}}_{01}$ and $\underline{\hat{A}}_{02}$:

$$\underline{\hat{A}}_0 \;=\; \underline{\hat{A}}_{01} + \underline{\hat{A}}_{02} \tag{C.6}$$

with

$$\underline{\hat{A}}_{01} \;=\; \Delta s\cdot\frac{H^2}{[1+f(\hat{p}_\sigma)]}\cdot\begin{pmatrix} 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\end{pmatrix} \tag{C.7a}$$

and

$$\underline{\hat{A}}_{02} \;=\; \Delta s \cdot \frac{H}{[1+f(\hat{p}_\sigma)]} \cdot \begin{pmatrix} 0 & 0 & +1 & 0 & \\ -(\hat{g}/H) & 0 & 0 & +1 & \\ -1 & 0 & 0 & 0 & \\ 0 & -1 & +(\hat{g}/H) & 0 & 0 \end{pmatrix} \; . \tag{C.7b}$$

Since

$$\hat{D}\,\underline{\hat{A}}_0 \;=\; \underline{\hat{A}}_0\,\hat{D} \;\Longrightarrow\; \hat{D}^{\nu}\,\vec{\hat{y}}_0 \;=\; \underline{\hat{A}}^{\nu}\vec{\hat{y}}_0 \;\Longrightarrow\; \left\{\exp\left[\hat{D}\right]\right\}\vec{\hat{y}}_0 \;=\; \left\{\exp\left[\underline{\hat{A}}_0\right]\right\}\vec{\hat{y}}_0$$

the transfer matrix for

$$\vec{\hat{y}}_0 \;=\; \begin{pmatrix} x \\ p_x \\ z \\ p \end{pmatrix} \tag{C.8}$$

reads as:

$$\underline{M}_0 \;=\; \exp\left[\underline{\hat{A}}_0\right] \; . \tag{C.9}$$

Using the equations

$$\underline{\hat{A}}_{01} \cdot \underline{\hat{A}}_{02} \;=\; \underline{\hat{A}}_{02} \cdot \underline{\hat{A}}_{01}$$

we have:

$$\exp\left[\underline{\hat{A}}\right] \;=\; \exp\left[\underline{\hat{A}}_{01}\right] \cdot \exp\left[\underline{\hat{A}}_{02}\right] \; .$$

Also since

$$\left[\underline{\hat{A}}_{01}\right]^{\nu} \;=\; \underline{0} \;\text{ for }\; \nu \;>\; 1$$

we get:

$$\exp\left[\underline{\hat{A}}_{01}\right] \;=\; \underline{1} + \underline{\hat{A}}_{01} \; .$$

Because

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -(\hat{g}/H) & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & +(\hat{g}/H) & 0 \end{pmatrix}^{2n} \;=\; (-1)^n \cdot \underline{1} \; ;$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -(\hat{g}/H) & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & +(\hat{g}/H) & 0 \end{pmatrix}^{2n+1} \;=\; (-1)^n \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ -(\hat{g}/H) & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & +(\hat{g}/H) & \end{pmatrix}$$

we obtain :

$$\exp\left[\underline{\hat{A}}_{02}\right] =$$

$$\sum_{n,=0}^{\infty} \frac{1}{(2n)!} \cdot (-1)^n \cdot (\Delta\Theta)^{2n} \cdot \underline{1}$$

$$+ \sum_{n,=0}^{\infty} \frac{1}{(2n+1)!} \cdot (-1)^n \cdot (\Delta\Theta)^{2n+1} \cdot \begin{pmatrix} 0 & 0 & +1 & 0 \\ -(\hat{g}/H) & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & +(\hat{g}/H) & 0 \end{pmatrix}$$

$$= \underline{1} \cdot \cos(\Delta\Theta) + \begin{pmatrix} 0 & 0 & +1 & 0 \\ -(\hat{g}/H) & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & +(\hat{g}/H) & 0 \end{pmatrix} \cdot \sin(\Delta\Theta)$$

$$= \begin{pmatrix} \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) & 0 \\ -(\hat{g}/H) \cdot \sin(\Delta\Theta) & \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) \\ -\sin(\Delta\Theta) & 0 & \cos(\Delta\Theta) & 0 \\ 0 & -\sin(\Delta\Theta) & +(\hat{g}/H) \cdot \sin(\Delta\Theta) & \cos(\Delta\Theta) \end{pmatrix}$$

with

$$\Delta\Theta = \frac{H(s_0) \cdot \Delta s}{[1 + f(\hat{p}_\sigma)]} .$$

Then from (C.9) :

$$\underline{M}_0 = \left[\underline{1} + \underline{\hat{A}}_{01}\right]$$

$$\times \begin{pmatrix} \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) & 0 \\ -(\hat{g}/H) \cdot \sin(\Delta\Theta) & \cos(\Delta\Theta) & 0 & +\sin(\Delta\Theta) \\ -\sin(\Delta\Theta) & 0 & \cos(\Delta\Theta) & 0 \\ 0 & -\sin(\Delta\Theta) & +(\hat{g}/H) \cdot \sin(\Delta\Theta) & \cos(\Delta\Theta) \end{pmatrix} . \quad \text{(C.10)}$$

For the variable σ we obtain from (C.3b) :

$$\hat{D}^2\,\hat{\sigma} = \hat{D}\,F_5(\vec{\hat{y}})$$

$$= \left\{F_1(\vec{\hat{y}}) \cdot \frac{\partial}{\partial\hat{y}_1} + F_2(\vec{\hat{y}}) \cdot \frac{\partial}{\partial\hat{y}_2} + F_3(\vec{\hat{y}}) \cdot \frac{\partial}{\partial\hat{y}_3} + F_4(\vec{\hat{y}}) \cdot \frac{\partial}{\partial\hat{y}_4}\right\} F_5(\vec{\hat{y}})$$

$$= \frac{H \cdot \Delta s}{[1 + f(\hat{p}_\sigma)]} \left\{\hat{y}_3 \cdot \frac{\partial}{\partial\hat{y}_1} - [H \cdot \hat{y}_1 - \hat{y}_4] \cdot \frac{\partial}{\partial\hat{y}_2} - \hat{y}_1 \cdot \frac{\partial}{\partial\hat{y}_3} - [H \cdot \hat{y}_3 + \hat{y}_2] \cdot \frac{\partial}{\partial\hat{y}_4}\right.$$

$$\left. -(\hat{g}/H) \cdot \hat{y}_1 \cdot \frac{\partial}{\partial\hat{y}_2} + (\hat{g}/H) \cdot \hat{y}_3 \cdot \frac{\partial}{\partial\hat{y}_4}\right\}$$

$$\left\{\frac{(-H) \cdot f'(\hat{p}_\sigma) \cdot \Delta s}{[1 + f(\hat{p}_\sigma)]^2} \left[\frac{1}{2} H \cdot (\hat{y}_1^2 + \hat{y}_3^2) + (\hat{y}_2 \cdot \hat{y}_3 - \hat{y}_4 \cdot \hat{y}_1)\right]\right\}$$

$$
\begin{aligned}
&= \frac{H \cdot \Delta s}{[1+f(\hat{p}_\sigma)]} \cdot \frac{(-H)\cdot f'(\hat{p}_\sigma)\cdot \Delta s}{[1+f(\hat{p}_\sigma)]^2} \\
&\quad \times \{\hat{y}_3 \cdot [H\cdot\hat{y}_1 - \hat{y}_4] - [H\cdot\hat{y}_1 - \hat{y}_4]\cdot\hat{y}_3 - \hat{y}_1\cdot[H\cdot\hat{y}_3+\hat{y}_2] + [H\cdot\hat{y}_3+\hat{y}_2]\cdot\hat{y}_1 \\
&\qquad\qquad -(\hat{g}/H)\cdot[\hat{y}_1\cdot\hat{y}_3 + \hat{y}_3\cdot\hat{y}_1]\} \\
&= -2\,\frac{H\cdot\Delta s}{[1+f(\hat{p}_\sigma)]}\cdot\frac{(-H)\cdot f'(\hat{p}_\sigma)\cdot\Delta s}{[1+f(\hat{p}_\sigma)]^2}\cdot(\hat{g}/H)\cdot\hat{y}_1\cdot\hat{y}_3 \\
&= 2\cdot[\Delta\Theta]^2\cdot\frac{f'(\hat{p}_\sigma)\cdot\Delta s}{[1+f(\hat{p}_\sigma)]}\cdot(\hat{g}/H)\cdot\hat{y}_1\cdot\hat{y}_3\ .
\end{aligned}
$$

Furthermore, using the relations :

$$
\begin{aligned}
\hat{D}^{2n}\,[\hat{y}_1\,\hat{y}_3] &= (-1)^n\cdot[2\cdot\Delta\Theta]^{2n}\cdot[\hat{y}_1\,\hat{y}_3]\ ; \\
\hat{D}^{2n+1}\,[\hat{y}_1\,\hat{y}_3] &= (-1)^n\cdot[2\cdot\Delta\Theta]^{2n+1}\cdot\frac{1}{2}\,[\hat{y}_3^2-\hat{y}_1^2]\ ;
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow\quad \left\{\exp[\hat{D}]\right\}[\hat{y}_1\,\hat{y}_3] &= \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\hat{D}^n\,[\hat{y}_1\,\hat{y}_3] \\
&= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\cdot\hat{D}^{2n}\,[\hat{y}_1\,\hat{y}_3] + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\cdot\hat{D}^{2n+1}\,[\hat{y}_1\,\hat{y}_3] \\
&= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\cdot(-1)^n\cdot[2\cdot\Delta\Theta]^{2n}\cdot[\hat{y}_1\,\hat{y}_3] \\
&\quad + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\cdot(-1)^n\cdot[2\cdot\Delta\Theta]^{2n+1}\cdot\frac{1}{2}\,[\hat{y}_3^2-\hat{y}_1^2] \\
&= [\hat{y}_1\,\hat{y}_3]\cdot\cos[2\,\Delta\Theta] + \frac{1}{2}\,[\hat{y}_3^2-\hat{y}_1^2]\cdot\sin[2\,\Delta\Theta]
\end{aligned}
$$

and

$$
\hat{D}\,F_5(\vec{y}) = \kappa\cdot\hat{D}^2[\hat{y}_1\,\hat{y}_3]
$$

with

$$
\begin{aligned}
\kappa &= -\frac{1}{2}\,\frac{f'(p_\sigma)}{[1+f(p_\sigma)]^2}\cdot\frac{\hat{g}}{H} \\
&= -\frac{1}{2}\,\frac{f'(p_\sigma)}{[1+f(p_\sigma)]}\cdot\frac{g}{H}\ ,
\end{aligned}
$$

we get :

$$
\begin{aligned}
\left\{\exp\left[\hat{D}\right]\right\}\hat{\sigma} &= \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\hat{D}\,\hat{\sigma} \\
&= \hat{\sigma} + \hat{D}\,\hat{\sigma} + \sum_{n=2}^{\infty}\frac{1}{n!}\cdot\hat{D}^n\,\hat{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&= \hat{\sigma} + F_5(\vec{\hat{y}}) + \sum_{n=2}^{\infty} \frac{1}{n!} \cdot \hat{D}^{n-1} F_5(\vec{\hat{y}}) \\
&= \hat{\sigma} + F_5(\vec{\hat{y}}) + \sum_{n=2}^{\infty} \frac{1}{n!} \cdot \hat{D}^{n-2} \cdot \hat{D} F_5(\vec{\hat{y}}) \\
&= \hat{\sigma} + F_5(\vec{\hat{y}}) + \kappa \cdot \sum_{n=2}^{\infty} \frac{1}{n!} \cdot \hat{D}^{n} [\hat{y}_1 \hat{y}_3] \\
&= \hat{\sigma} + F_5(\vec{\hat{y}}) + \kappa \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \hat{D}^{n} [\hat{y}_1 \hat{y}_3] - \kappa \cdot (1 + \hat{D}) [\hat{y}_1 \hat{y}_3] \\
&= \hat{\sigma} + F_5(\vec{\hat{y}}) - \kappa \cdot [\hat{y}_1 \hat{y}_3] - \kappa \cdot [2 \cdot \Delta\Theta] \cdot \frac{1}{2} [\hat{y}_3^2 - \hat{y}_1^2] \\
&\qquad + \kappa \cdot [\hat{y}_1 \hat{y}_3] \cdot \cos[2 \cdot \Delta\Theta] + \kappa \cdot \frac{1}{2} [\hat{y}_3^2 - \hat{y}_1^2] \cdot \sin[2 \cdot \Delta\Theta] \\
&= \hat{\sigma} + F_5(\vec{\hat{y}}) - \kappa \cdot [\hat{y}_1 \hat{y}_3] \cdot \{1 - \cos[2 \cdot \Delta\Theta]\} \\
&\qquad + \kappa \cdot \frac{1}{2} [\hat{y}_3^2 - \hat{y}_1^2] \cdot \{\sin[2 \cdot \Delta\Theta] - [2 \cdot \Delta\Theta]\} \ . \qquad \text{(C.11)}
\end{aligned}
$$

C.2 Thin - Lens Transport Map

Equations (C.9), (C.11) and (C.3c) finally lead to:

$$\vec{y}_0^{\,f} = \underline{M}_0 \, \vec{y}_0^{\,i} \ ; \qquad \text{(C.12a)}$$

$$
\begin{aligned}
\sigma^f &= \sigma^i - \frac{f'(p_\sigma^i)}{[1 + f(p_\sigma^i)]} \cdot \Delta\Theta \\
&\quad \times \left\{ \frac{1}{2} H(s_0) \cdot \left[(x^i)^2 + (z^i)^2 \right] + \left[p_x^i \cdot z^i - p_z^i \cdot x^i \right] \right\} \\
&\quad - \kappa \cdot [x^i \cdot z^i] \cdot \{1 - \cos[2 \cdot \Delta\Theta]\} \\
&\quad + \kappa \cdot \frac{1}{2} [(z^i)^2 - (x^i)^2] \cdot \{\sin[2 \cdot \Delta\Theta] - [2 \cdot \Delta\Theta]\} \ ; \qquad \text{(C.12b)}
\end{aligned}
$$

$$p_\sigma^f = p_\sigma^i \qquad \text{(C.12c)}$$

with

$$\Delta\Theta = \frac{H(s_0) \cdot \Delta s}{[1 + f(p_\sigma^i)]} \qquad \text{(C.12d)}$$

$$\kappa = -\frac{1}{2} \frac{f'(p_\sigma^i)}{[1 + f(p_\sigma^i)]} \cdot \frac{g}{H} \ , \qquad \text{(C.12e)}$$

and $\underline{M}_0$ given by (C.10), where the matrix $\underline{\hat{A}}_{01}$ appearing in (C.10) takes the form:

$$\underline{\hat{A}}_{01} = \Delta\Theta \cdot H(s_0) \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \text{(C.12f)}$$

(see eqn. (C.7a)).

Equation (C.12) contains as special cases the transport maps of a simple solenoid and a simple quadrupole already derived in section 4.

References

[1] G. Ripken, F. Schmidt: "A Sympletic Six-Dimensional Thin-Lens Formalism for Tracking"; DESY 95–63 and CERN/SL/95–12(AP), (1995)
and literature cited therein.

[2] F. Schmidt: "SIXTRACK version 1.2: Single particle tracking code treating transverse motion with synchrotron oscillations in a symplectic manner: User's reference manual"; CERN/SL/94–56 (AP), Sep 1994, 50pp.
and literature cited therein.
The updated manual can be retrieved from the WWW at the location: *http://hpariel.cern.ch/frs/Documentation/six.ps.*

[3] W. Gröbner: "Differentialgleichungen";
Erster Teil
Gewöhnliche Differentialgleichungen;
BI-Wissenschaftsverlag (1977).

[4] A.J. Dragt: "Lectures on Nonlinear Orbit Dynamics".
In Batavia 1981, Proceedings, Physics of High Energy Particle Accelerators; p. 147-313.

[5] D.P. Barber, K. Heinemann, K. Karantzouliss, H. Mais, G. Ripken: "The Influence of Damping in Nonlinear Fields on Electron Motion in Storage Rings"; DESY M–94–09, (1994).