

NAG C Library Chapter Introduction

c02 – Zeros of Polynomials

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1 Scope of the Chapter

This chapter is concerned with computing the zeros of a polynomial with real or complex coefficients.

2 Background

Let $f(z)$ be a polynomial of degree n with complex coefficients a_i :

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n, \quad a_0 \neq 0.$$

A complex number z_1 is called a *zero* of $f(z)$ (or equivalently a *root* of the equation $f(z) = 0$), if:

$$f(z_1) = 0.$$

If z_1 is a zero, then $f(z)$ can be divided by a factor $(z - z_1)$:

$$f(z) = (z - z_1)f_1(z)$$

where $f_1(z)$ is a polynomial of degree $n - 1$. By the Fundamental Theorem of Algebra, a polynomial $f(z)$ always has a zero, and so the process of dividing out factors $(z - z_i)$ can be continued until we have a complete *factorization* of $f(z)$

$$f(z) \equiv a_0(z - z_1)(z - z_2) \dots (z - z_n).$$

Here the complex numbers z_1, z_2, \dots, z_n are the zeros of $f(z)$; they may not all be distinct, so it is sometimes more convenient to write:

$$f(z) \equiv a_0(z - z_1)^{m_1}(z - z_2)^{m_2} \dots (z - z_k)^{m_k}, \quad k \leq n,$$

with distinct zeros z_1, z_2, \dots, z_k and multiplicities $m_i \geq 1$. If $m_i = 1$, z_i is called a *single zero*, if $m_i > 1$, z_i is called a *multiple* or *repeated zero*; a multiple zero is also a zero of the derivative of $f(z)$.

If the coefficients of $f(z)$ are all real, then the zeros of $f(z)$ are either real or else occur as pairs of complex conjugate numbers $x + iy$ and $x - iy$. A pair of complex conjugate zeros are the zeros of a quadratic factor of $f(z)$, $(z^2 + rz + s)$, with real coefficients r and s .

A zero is called *ill-conditioned* if it is sensitive to small changes in the coefficients of $f(z)$. Conversely, a zero is called *well-conditioned* if it is comparatively insensitive to such perturbations. Roughly speaking, a zero which is well separated from other zeros is well-conditioned, while zeros which are close together are ill-conditioned, but in talking about ‘closeness’ the decisive factor is not the absolute distance between neighbouring zeros but their *ratio*: if the ratio is close to 1 the zeros are ill-conditioned. In particular, multiple zeros are ill-conditioned. A multiple zero is usually split into a cluster of zeros by perturbations in the polynomial or computational inaccuracies.

The accuracy of the roots will depend on how ill-conditioned they are. Peters and Wilkinson (1970) describe techniques for estimating the errors in the zeros after they have been computed.

3 References

Peters G and Wilkinson J H (1970) The least-squares problem and pseudo-inverses *Comput. J.* **13** 309–316

4 Available Functions

c02afc Zeros of a polynomial with complex coefficients
 c02agc Zeros of a polynomial with real coefficients
 c02akc Zeros of a cubic polynomial with real coefficients
 c02alc Zeros of a real quartic polynomial with real coefficients
