Introduction to
Chiral Perturbation Theory

H. Leutwyler
University of Bern

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I. Standard Model at low energies

1. Interactions

Local symmetries

2. QED+QCD

Precision theory for $E \ll 100\,\text{GeV}$

Qualitative difference QED $\Longleftrightarrow$ QCD

3. Chiral symmetry

Some of the quarks happen to be light

Approximate chiral symmetry

Spontaneous symmetry breakdown

4. Goldstone theorem

If N_f of the quark masses are put equal to zero

QCD contains N_f^2-1 Nambu-Goldstone bosons

5. Gell-Mann-Oakes-Renner relation

Quark masses break chiral symmetry

NGBs pick up mass

M_π^2 is proportional to $m_u + m_d$

II. Chiral perturbation theory

6. Group geometry

Symmetry group of the Hamiltonian G

Symmetry group of the ground state H

Nambu-Goldstone bosons live on G/H

7. Generating functional of QCD

Collects the Green functions of the theory

8. Ward identities

Symmetries of the generating functional

9. Low energy expansion

Taylor series in powers of external momenta

NGBs generate infrared singularities

10. Effective Lagrangian

Singularities due to the Nambu-Goldstone bosons can be worked out with an effective field theory. Side remark: for nonrelativistic systems, there is a complication. In that case, $\mathcal{L}_{eff}$ is in general invariant only up to a total derivative.

11. Explicit construction of $\mathcal{L}_{eff}$

III. Illustrations

12. Some tree level calculations

Leading terms of the chiral perturbation series for the quark condensate and for M_π, F_π

13. M_π beyond tree level

Contributions to M_π at NL and NNL orders

14. F_π to one loop

Chiral logarithm in F_π, low energy theorem for scalar radius

15. Pion form factors

Charge radius of the pion, scalar radius
Dispersion relations

16. Lattice results for M_π, F_π

Determination of the effective coupling constants ℓ_3, ℓ_4 on the lattice

17. $\pi\pi$ scattering

χPT , lattice, precision experiments

18. Conclusions for SU(2)×SU(2)

19. Expansion in powers of m_s

Convergence, validity of Zweig rule

20. Conclusions for SU(3)×SU(3)

IV. Some recent results

21. Masses of the light quarks

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I. Standard Model at low energies

1. Interactions

strong weak e.m. gravity

SU(3) × SU(2) × U(1) × D

Gravity

understood only at classical level
gravitational waves √
quantum theory of gravity ?
classical theory adequate for

$$r \gg \sqrt{\frac{G\,\hbar}{c^3}} = 1.6 \cdot 10^{-35}\,\mathrm{m}$$

Weak interaction

frozen at low energies

$$E \ll M_{\mathrm{W}}\, c^2 \simeq 80\,\mathrm{GeV}$$

⇒ structure of matter: only strong and electromagnetic interaction

⇒ neutrini decouple

Electromagnetic interaction

Maxwell $\sim$ 1860

survived relativity and quantum theory, unharmed

- Electrons in electromagnetic field ($\hbar = c = 1$)

$$\frac{1}{i}\frac{\partial\psi}{\partial t} - \frac{1}{2m_e^2}(\vec{\nabla} + i\,e\vec{A})^2\psi - e\,\varphi\,\psi = 0$$

contains the potentials $\vec{A}$, φ

- only $\quad \vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial\vec{A}}{\partial t} \quad$ and $\quad \vec{B} = \vec{\nabla}\times\vec{A}$

are of physical significance

- Schrödinger equation is invariant under gauge transformations

$$\vec{A}\,' = \vec{A} + \vec{\nabla}f \ , \qquad \varphi\,' = \varphi - \frac{\partial f}{\partial t} \ , \qquad \psi\,' = e^{-ief}\,\psi$$

describe the same physical situation as $\vec{A}, \varphi, \psi$

- Equivalence principle of the e.m. interaction:

$$\boxed{\psi \text{ physically equivalent to } e^{-ief}\,\psi}$$

- e^{-ief} is unitary 1×1 matrix, $e^{-ief} \in$ U(1)
  $f = f(\vec{x}, t)$ space-time dependent function

- gauge invariance $\Longleftrightarrow$ local U(1) symmetry
  electromagnetic field is gauge field of U(1)
  Weyl 1929

- U(1) symmetry $+$ renormalizability
  fully determine the e.m. interaction

Strong interaction

nuclei = p + n ∼ 1930

- Nuclear forces

  Yukawa ∼ 1935

$$V_{e.m.} = -\frac{e^2}{4\pi r} \qquad V_s = -\frac{h^2}{4\pi r}\, e^{-\frac{r}{r_0}}$$

$$\frac{e^2}{4\pi} \simeq \frac{1}{137} \qquad \frac{h^2}{4\pi} \simeq 13$$

long range | short range

$$r_0 = \infty \qquad r_0 = \frac{\hbar}{M_\pi c} = 1.4 \cdot 10^{-15}\,\mathrm{m}$$

$$M_\gamma = 0 \qquad M_\pi\, c^2 \simeq 140\,\mathrm{MeV}$$

- Problem with Yukawa formula:

  p and n are extended objects

  diameter comparable to range of force

  formula only holds for $r \gg$ diameter

- Protons, neutrons composed of quarks

p $= uud$ n $= udd$

- Quarks carry internal quantum number

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

occur in 3 “colours”

- Strong interaction is invariant under local rotations in colour space 1973

$$u' = U \cdot u \qquad d' = U \cdot d$$

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \in \mathrm{SU(3)}$$

- Can only be so if the strong interaction is also mediated by a gauge field

gauge field of SU(3) $\Longrightarrow$ strong interaction

Quantum chromodynamics

Comparison of e.m. and strong interaction

	QED	QCD
symmetry	U(1)	SU(3)
gauge field	$\vec{A}, \varphi$	gluon field
particles	photons	gluons
source	charge	colour
coupling constant	e	g

- All charged particles generate e.m. field
- All coloured particles generate gluon field
- Leptons do not interact strongly because they do not carry colour
- Equivalence principle of the strong interaction:

$$U \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ physically equivalent to } \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

2. QED+QCD

Effective theory for $E \ll M_{\rm W}c^2 \simeq 80\,\text{GeV}$

Symmetry	U(1) × SU(3)
Lagrangian	QED+QCD

- Dynamical variables:
  gauge fields for photons and gluons
  Fermi fields for leptons and quarks

- Interaction fully determined by group geometry
  Lagrangian contains 2 coupling constants

$$\boxed{e,\ g}$$

- Quark and lepton mass matrices can be brought to diagonal form, eigenvalues real, positive

$$\boxed{m_e,\ m_\mu,\ m_\tau,\ m_u,\ m_d,\ m_s,\ m_c,\ m_b,\ m_t}$$

- Transformation generates vacuum angle

$$\boxed{\theta}$$

- Precision theory for cold matter, atomic structure, solids, ...

  Bohr radius: $$a = \frac{4\pi}{e^2\, m_e}$$

- θ breaks CP

  Neutron dipole moment is very small

⇒ strong upper limit, $\theta \simeq 0$

Qualitative difference between e.m. and strong interactions

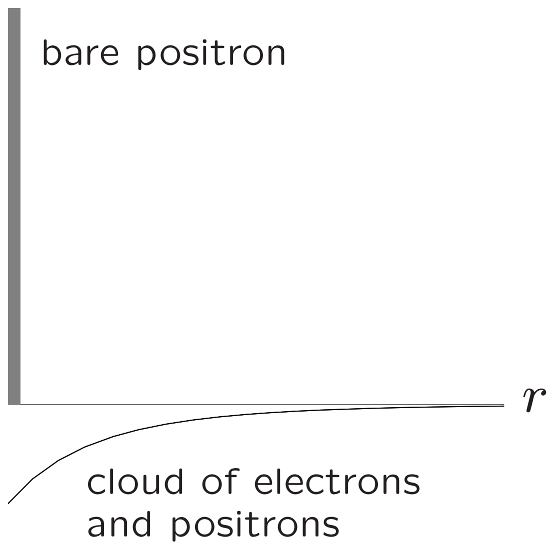
- Photons do not have charge
- Gluons do have colour

$x_1 \cdot x_2 = x_2 \cdot x_1$ for $x_1, x_2 \in U(1)$ abelian

$x_1 \cdot x_2 \neq x_2 \cdot x_1$ for $x_1, x_2 \in SU(3)$

⇒ Consequence for vacuum polarization

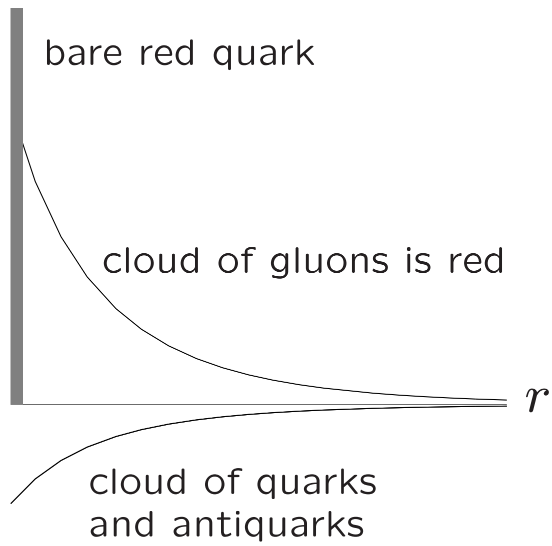
QED
Density of charge



$e < e_{\text{bare}}$

vacuum
shields charge

QCD
Density of colour



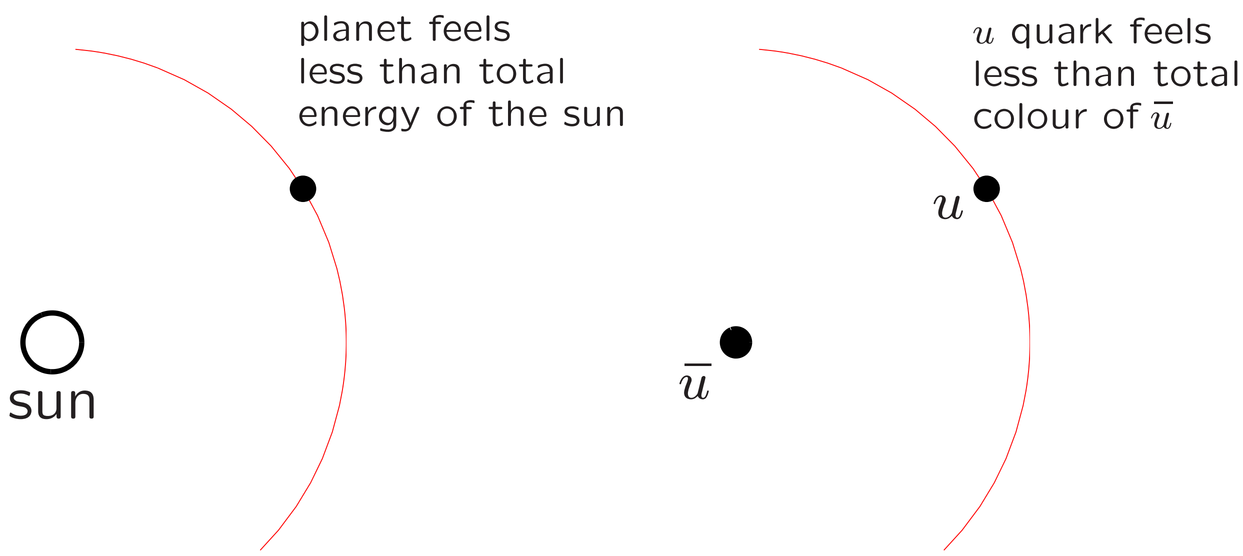
$g > g_{\text{bare}}$

vacuum
amplifies colour

Comparison with gravity

- source of gravitational field: energy
  gravitational field <u>does</u> carry energy
- source of e.m. field: charge
  e.m. field <u>does not</u> carry charge
- source of gluon field: colour
  gluon field <u>does</u> carry colour

gravity **strong interaction**



Perihelion shift of Mercury:

$$43'' = 50'' - 7'' \text{ per century}$$
$$\Uparrow$$

- Force between u and $\bar{u}$:

$$V_s = -\frac{4}{3}\frac{g^2}{4\pi r}\,, \qquad g \to 0 \quad \text{for} \quad r \to 0$$

$$\frac{g^2}{4\pi} = \frac{6\pi}{(11N_c - 2N_f)\,|\ln(r\,\Lambda_{\rm QCD})|}$$

$$|\ln(r\,\Lambda_{\rm QCD})| \simeq 7 \quad \text{for} \;\; r = \frac{\hbar}{M_Z\,c} \simeq 2\cdot 10^{-18}\,\text{m}$$

- Vacuum amplifies gluonic field of a bare quark
- Field energy surrounding isolated quark $= \infty$
  Only colour neutral states have finite energy

⇒ Confinement of colour

- Theoretical evidence for confinement meagre
  Experimental evidence much more convincing

QED: interaction weak at low energies
QCD: interaction strong at low energies

$\frac{e^2}{4\pi} \simeq \frac{1}{137}$	$\frac{g^2}{4\pi} \simeq 1$
photons, leptons nearly decouple	gluons, quarks confined

- Nuclear forces = van der Waals forces of QCD

3. Chiral symmetry

- Photons are extremely useful to probe QCD

Much of what we know about the structure of the hadrons stems scattering experiments involving electrons or photons

$e+N \rightarrow e+N$ form factors of the nucleon

$e+N \rightarrow e+hadrons$ deep inelastic scattering

electroproduction, photoproduction

$\Rightarrow$ several lectures and seminars at this school

- For bound states of quarks,
  e.m. interaction is a small perturbation

  Perturbation series in powers of $\frac{e^2}{4\pi}$ $\surd$

  Discuss only the leading term: set $e = 0$

- Lagrangian then reduces to QCD

$$\boxed{\; g\,,\, m_u\,, m_d\,,\, m_s\,,\, m_c\,,\, m_b\,,\, m_t \;}$$

- m_u, m_d, m_s happen to be light

  Consequence:
  Approximate flavour symmetries
  Play a crucial role for the low energy properties

Theoretical paradise

$m_u = m_d = m_s = 0$

$m_c = m_b = m_t = \infty$

QCD with 3 massless quarks

- Lagrangian contains a single parameter: g
  g is net colour of a quark
  depends on radius of the region considered

- Colour contained within radius r

$$\frac{g^2}{4\pi} = \frac{2\pi}{9\,|\ln(r\,\Lambda_{\text{QCD}})|}$$

- Intrinsic scale Λ_{QCD} is meaningful,
  but not dimensionless

⇒ No dimensionless free parameter

All dimensionless physical quantities are pure numbers, determined by the theory
Cross sections can be expressed in terms of Λ_{QCD} or in the mass of the proton

- Interactions of u, d, s are identical
  If the masses are set equal to zero,
  there is no difference at all

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- Lagrangian symmetric under $u \leftrightarrow d \leftrightarrow s$

$$q' = V \cdot q \qquad V \in \mathrm{SU(3)}$$

  V acts on quark flavour, mixes u, d, s

- More symmetry: For massless fermions,
  right and left do not communicate

⇒ Lagrangian of massless QCD is invariant under independent rotations of the right– and left–handed quark fields

$$q_{\mathrm{R}} = \tfrac{1}{2}(1 + \gamma_5)\, q \,, \qquad q_{\mathrm{L}} = \tfrac{1}{2}(1 - \gamma_5)\, q$$

$$q'_{\mathrm{R}} = V_{\mathrm{R}} \cdot q_{\mathrm{R}} \qquad\qquad q'_{\mathrm{L}} = V_{\mathrm{L}} \cdot q_{\mathrm{L}}$$

$$\mathrm{SU(3)_R} \times \mathrm{SU(3)_L}$$

- Massless QCD invariant under $\text{SU}(3)_\text{R} \times \text{SU}(3)_\text{L}$

  SU(3) has 8 parameters

⇒ Symmetry under Lie group with 16 parameters

⇒ 16 conserved "charges"

$Q_1^\text{V}, \ldots, Q_8^\text{V}$ (vector currents, $R+L$)

$Q_1^\text{A}, \ldots, Q_8^\text{A}$ (axial currents, $R-L$)

commute with the Hamiltonian:

$$[Q_i^\text{V}, H_0] = 0 \qquad [Q_i^\text{A}, H_0] = 0$$

"Chiral symmetry" of massless QCD

- Vafa and Witten 1984: state of lowest energy is invariant under the vector charges

  $Q_i^\text{V}\,|0\rangle = 0$

- Axial charges ? $Q_i^\text{A}\,|0\rangle = ?$

Two alternatives for axial charges

$Q_i^{\mathsf{A}}\,|0\rangle = 0$
Wigner-Weyl realization of G ground state is symmetric $\langle 0|\,\overline{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,|0\rangle = 0$ ordinary symmetry spectrum contains parity partners degenerate multiplets of G

$Q_i^{\mathsf{A}}\,|0\rangle \neq 0$
Nambu-Goldstone realization of G ground state is asymmetric $\langle 0|\,\overline{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,|0\rangle \neq 0$ "order parameter" spontaneously broken symmetry spectrum contains Nambu-Goldstone bosons degenerate multiplets of $\mathsf{SU(3)_V} \subset \mathsf{G}$

$$\mathsf{G} = \mathsf{SU(3)_R} \times \mathsf{SU(3)_L}$$

- Spontaneous symmetry breakdown was discovered in condensed matter physics:

  Spontaneous magnetization selects direction

$\Rightarrow$ Rotation symmetry is spontaneously broken

  Nambu-Goldstone bosons = spin waves, magnons

- Nambu 1960: state of lowest energy in particle physics is not invariant under chiral rotations

  $$Q_i^{\mathsf{A}}\,|0\rangle \neq 0$$

  For dynamical reasons, the state of lowest energy must be asymmetric

$\Rightarrow$ Chiral symmetry is spontaneously broken

- Very strong experimental evidence $\surd$
- Theoretical understanding on the basis of the QCD Lagrangian ?

- Analog of Magnetization ?

$$\bar{q}_{\mathsf{R}}\, q_{\mathsf{L}} = \begin{pmatrix} \bar{u}_{\mathsf{R}}\, u_{\mathsf{L}} & \bar{d}_{\mathsf{R}}\, u_{\mathsf{L}} & \bar{s}_{\mathsf{R}}\, u_{\mathsf{L}} \\ \bar{u}_{\mathsf{R}}\, d_{\mathsf{L}} & \bar{d}_{\mathsf{R}}\, d_{\mathsf{L}} & \bar{s}_{\mathsf{R}}\, d_{\mathsf{L}} \\ \bar{u}_{\mathsf{R}}\, s_{\mathsf{L}} & \bar{d}_{\mathsf{R}}\, s_{\mathsf{L}} & \bar{s}_{\mathsf{R}}\, s_{\mathsf{L}} \end{pmatrix}$$

Transforms like $(\bar{3}, 3)$ under $\mathrm{SU}(3)_{\mathsf{R}} \times \mathrm{SU}(3)_{\mathsf{L}}$

If the ground state were symmetric, the matrix $\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle$ would have to vanish, because it singles out a direction in flavour space

"quark condensate", is quantitative measure of spontaneous symmetry breaking
"order parameter"

$\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle \Leftrightarrow$ magnetization

- Ground state is invariant under $\mathrm{SU}(3)_{\mathsf{V}}$

$\Rightarrow$ $\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle$ is proportional to unit matrix

$\langle 0|\, \bar{u}_{\mathsf{R}}\, u_{\mathsf{L}}\, |0\rangle = \langle 0|\, \bar{d}_{\mathsf{R}}\, d_{\mathsf{L}}\, |0\rangle = \langle 0|\, \bar{s}_{\mathsf{R}}\, s_{\mathsf{L}}\, |0\rangle$

$\langle 0|\, \bar{u}_{\mathsf{R}}\, d_{\mathsf{L}}\, |0\rangle = \ldots = 0$

4. Goldstone Theorem

- Consequence of $Q_i^{\mathsf{A}}\,|0\rangle \neq 0$:

$$H_0\,Q_i^{\mathsf{A}}\,|0\rangle = Q_i^{\mathsf{A}}\,H_0\,|0\rangle = 0$$

spectrum must contain 8 states

$$Q_1^{\mathsf{A}}\,|0\rangle, \ldots, Q_8^{\mathsf{A}}\,|0\rangle \qquad \text{with} \quad E = 0,$$

spin 0, negative parity, octet of $\mathrm{SU}(3)_{\mathrm{V}}$
Nambu-Goldstone bosons

- Argument is not water tight:

$$\langle 0|\,Q_i^{\mathsf{A}}\,Q_k^{\mathsf{A}}\,|0\rangle = \int d^3x d^3y\,\langle 0|\,A_i^0(x)\,A_k^0(y)\,|0\rangle$$

$\langle 0|\,A_i^0(x)\,A_k^0(y)\,|0\rangle$ only depends on $\vec{x}-\vec{y}$

$\Rightarrow$ $\langle 0|\,Q_i^{\mathsf{A}}\,Q_k^{\mathsf{A}}\,|0\rangle$ is proportional to the volume of the universe, $|Q_i^{\mathsf{A}}\,|0\rangle| = \infty$

- Rigorous version of Goldstone theorem: $\langle 0|\, \bar{q}_{\mathrm{R}}\, q_{\mathrm{L}}\, |0\rangle \neq 0 \Rightarrow \exists$ massless particles

Proof fasten seatbelts: takes 3 slides

$$Q = \int d^3x \, \bar{u}\gamma^0\gamma_5 d$$
$$[Q, \bar{d}\gamma_5 u] = -\bar{u}u - \bar{d}d$$

- $F^\mu(x-y) \equiv \langle 0|\, \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(y)\gamma_5 u(y)\, |0\rangle$
  Lorentz invariance $\Rightarrow$ $F^\mu(z) = z^\mu f(z^2)$
  Chiral symmetry $\Rightarrow$ $\partial_\mu F^\mu(z) = 0$

$$F^\mu(z) = \frac{z^\mu}{z^4} \times \text{constant (for } z^2 \neq 0)$$

- Spectral decomposition:

$$F^\mu(x-y) = \langle 0|\, \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(y)\gamma_5 u(y)\, |0\rangle$$

$$= \sum_n \langle 0|\, \bar{u}\gamma^\mu\gamma_5 d |n\rangle\langle n|\, \bar{d}\gamma_5 u\, |0\rangle\; e^{-i\, p_n(x-y)}$$

$p_n^0 \geq 0 \Rightarrow F^\mu(z)$ is analytic in z^0 for $\mathrm{Im}\, z^0 < 0$

$$F^\mu(z) = \frac{z^\mu}{\{(z^0 - i\epsilon)^2 - \vec{z}^{\,2}\}^2} \times \text{constant}$$

- Positive frequency part of massless propagator: (exercise # 1)

$$\Delta^+(z,0) = \frac{i}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ipz} \quad , \qquad p^0 = |\vec{p}|$$
$$= \frac{1}{4\pi\, i \, \{(z^0 - i\epsilon)^2 - \vec{z}^{\,2}\}}$$

- Result

$$\boxed{\langle 0|\, \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu \Delta^+(z,0)}$$

- Compare Källen–Lehmann representation:

$$\langle 0|\, \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(y)\gamma_5 u(y)\,|0\rangle$$
$$= (2\pi)^{-3} \int d^4p\, p^\mu\, \rho(p^2) e^{-ip(x-y)}$$
$$= \int_0^\infty ds\, \rho(s)\partial^\mu \Delta^+(x-y,s)$$

$\Delta^+(z,s) \Longleftrightarrow$ massive propagator

$$\Delta^+(z,s) = \frac{i}{(2\pi)^3} \int d^4p\, \theta(p^0)\, \delta(p^2 - s)\, e^{-ipz}$$

$\Rightarrow$ Only massless intermediate states contribute:

$$\rho(s) = C\,\delta(s)$$

- Why only massless intermediate states ?
  $\langle n|\bar{d}\gamma_5 u\,|0\rangle \neq 0$ only if $\langle n|$ has spin 0
  If $|n\rangle$ has spin 0 $\Rightarrow \langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)|n\rangle \propto p^\mu\, e^{-ipx}$
  $\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = 0 \Rightarrow p^2 = 0$

$\Rightarrow$ Either $\exists$ massless particles or $C = 0$

- Claim: $\langle 0|\bar{q}_\mathrm{R}\, q_\mathrm{L}|0\rangle \neq 0 \Rightarrow C \neq 0$
  Lorentz invariance, chiral symmetry

$\Rightarrow \langle 0|\bar{d}(y)\gamma_5 u(y)\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,|0\rangle = C'\,\partial^\mu\Delta^-(z)$

$\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x), \bar{d}(y)\gamma_5 u(y)]\,|0\rangle$

$$= C\partial^\mu\Delta^+(z,0) - C'\partial^\mu\Delta^-(z,0)$$

- Causality: if $x-y$ is spacelike, then
  $\langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x), \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = 0$

$\Rightarrow C' = -C$

$\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x), \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C\partial^\mu\Delta(z,0)$

$\Rightarrow \langle 0|\,[\,Q, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C$

- $\langle 0|\,[\,Q, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = -\langle 0|\bar{u}u + \bar{d}d\,|0\rangle = C$
  Hence $\langle 0|\bar{u}u + \bar{d}d\,|0\rangle \neq 0$ implies $C \neq 0$ qed.

5. Gell-Mann-Oakes-Renner relation

Spectrum of QCD with 3 massless quarks must contain 8 massless physical particles, $J^P = 0^-$

- Indeed, the 8 lightest mesons do have these quantum numbers:

  $\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$

  But massless they are not, because

  $m_u\,,\, m_d\,,\, m_s \neq 0$

  Quark masses break chiral symmetry

- Chiral symmetry broken in two ways:

  spontaneously $\langle 0 | \,\overline{q}_\mathrm{R}\, q_\mathrm{L}\, | 0 \rangle \neq 0$

  explicitly $m_u\,,\, m_d\,,\, m_s \neq 0$

- H_{QCD} only has <u>approximate</u> symmetry, to the extent that m_u, m_d, m_s are small

$$H_{\text{QCD}} = H_0 + H_1$$
$$H_1 = \int d^3x \, \{m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s\}$$

- H_0 is Hamiltonian of the massless theory, invariant under $\text{SU}(3)_{\text{R}} \times \text{SU}(3)_{\text{L}}$

- H_1 breaks the symmetry, transforms with $(3, \bar{3}) \oplus (\bar{3}, 3)$

- For the low energy structure of QCD, the heavy quarks do not play an essential role:
  c, b, t are singlets under $\text{SU}(3)_{\text{R}} \times \text{SU}(3)_{\text{L}}$
  Can include the heavy quarks in H_0

- Nambu-Goldstone bosons are massless only if the symmetry is exact

Gell-Mann-Oakes-Renner formula:

$$\boxed{M_\pi^2 = (m_u + m_d) \times |\langle 0|\bar{u}\, u\,|0\rangle| \times \frac{1}{F_\pi^2}}$$ 1968

$\Uparrow$ explicit $\Uparrow$ spontaneous

Coefficient: decay constant F_π

- Why $M_\pi^2 \propto (m_u + m_d)$?

$$\langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)|\pi^-\rangle = i\,\sqrt{2}\,F_\pi\, p^\mu e^{-ip\cdot x}$$
$$\langle 0|\bar{u}(x)\, i\,\gamma_5 d(x)|\pi^-\rangle = \sqrt{2}\,G_\pi e^{-ip\cdot x}$$

  - Current conservation

$$\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = (m_u + m_d)\bar{u}\, i\,\gamma_5 d$$
$$\Rightarrow \sqrt{2}F_\pi\, p^2 = (m_u + m_d)\,\sqrt{2}G_\pi$$
$$p^2 = M_\pi^2$$

$$\Rightarrow \boxed{M_\pi^2 = (m_u + m_d)\,\frac{G_\pi}{F_\pi}}$$ exact

  - Expand in powers of m_u, m_d:

$$\frac{G_\pi}{F_\pi} = B + O(m)$$
$$\Rightarrow M_\pi^2 = (m_u + m_d)\,B + O(m^2)$$

- $M_\pi^2 = (m_u + m_d)\, B + O(m^2)$
- M_π disappears if the symmetry breaking is turned off, $m_u, m_d \to 0$ $\surd$
- Explains why the pseudoscalar mesons have very different masses

  $M_{K^+}^2 = (m_u + m_s)\, B + O(m^2)$

  $M_{K^-}^2 = (m_d + m_s)\, B + O(m^2)$

$\Rightarrow$ M_K^2 is about 13 times larger than M_π^2, because m_u, m_d happen to be small compared to m_s

- First order perturbation theory also yields

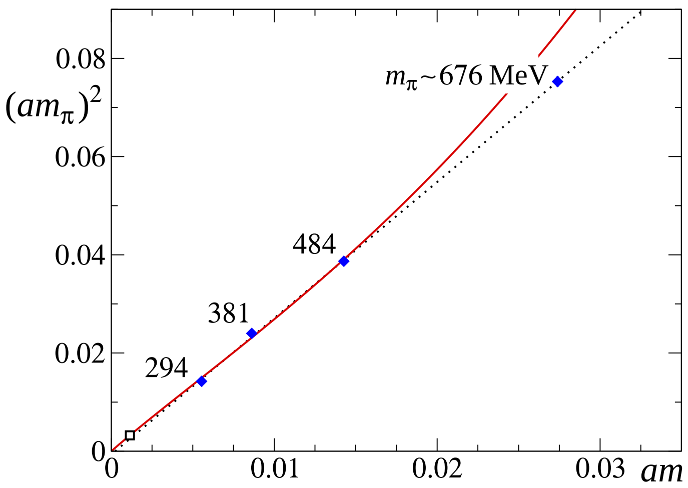
  $M_\eta^2 = \frac{1}{3}(m_u + m_d + 4m_s)B + O(m^2)$

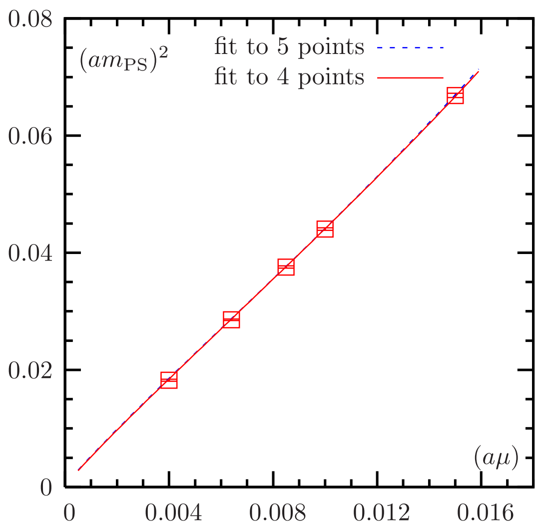
$\Rightarrow$ $M_\pi^2 - 4M_K^2 + 3M_\eta^2 = O(m^2)$

Gell-Mann-Okubo formula for M^2 $\surd$

Checking the GMOR formula on a lattice

- Can determine M_π as function of $m_u = m_d = m$





Lüscher, Lattice conference 2005 ETM collaboration, hep-lat/0701012

- No quenching, quark masses sufficiently light

⇒ Legitimate to use χPT for the extrapolation to the physical values of m_u, m_d

- Quality of data is impressive
- Proportionality of M_π^2 to the quark mass appears to hold out to values of m_u, m_d that are an order of magnitude larger than in nature
- Main limitation: systematic uncertainties

  in particular: $N_f = 2 \rightarrow N_f = 3$

II. Chiral perturbation theory

6. Group geometry

- QCD with 3 massless quarks:
  spontaneous symmetry breakdown
  from $\mathrm{SU(3)_R \times SU(3)_L}$ to $\mathrm{SU(3)_V}$
  generates 8 Nambu-Goldstone bosons

- Generalization: suppose symmetry group
  of Hamiltonian is Lie group G
  Generators $Q_1, Q_2, \ldots, Q_D$, $D = \mathrm{dim(G)}$
  For some generators $Q_i\,|0\rangle \neq 0$
  How many Nambu-Goldstone bosons ?

- Consider those elements of the Lie algebra
  $Q = \alpha_1 Q_1 + \ldots + \alpha_n Q_D$, for which $Q\,|0\rangle = 0$
  These elements form a subalgebra:
  $Q\,|0\rangle = 0,\ Q'\,|0\rangle = 0 \ \Rightarrow [Q, Q']\,|0\rangle = 0$
  Dimension of subalgebra: $d \leq D$

- Of the D vectors $Q_i\,|0\rangle$
  $D - d$ are linearly independent

$\Rightarrow$ $D - d$ different physical states of zero mass

$\Rightarrow$ $D - d$ Nambu-Goldstone bosons

- Subalgebra generates subgroup H $\subset$ G
  H is symmetry group of the ground state
  coset space G/H contains as many parameters as there are Nambu-Goldstone bosons
  $d = \text{dim(H)}$, $D = \text{dim(G)}$

$\Rightarrow$ Nambu-Goldstone bosons live on the coset G/H

- Example: QCD with N_f massless quarks
  G $= \text{SU}(N_f)_\text{R} \times \text{SU}(N_f)_\text{L}$
  H $= \text{SU}(N_f)_\text{V}$
  $D = 2\,(N_f^2 - 1)$, $d = N_f^2 - 1$
  $N_f^2 - 1$ Nambu-Goldstone bosons

- It so happens that $m_u, m_d \ll m_s$

- $m_u = m_d = 0$ is an excellent approximation
  SU(2)$_\text{R}\times$ SU(2)$_\text{L}$ is a nearly exact symmetry
  $N_f = 2$, $N_f^2 - 1 = 3$ Nambu-Goldstone bosons (pions)

7. Generating functional of QCD

- Basic objects for quantitative analysis of QCD: Green functions of the currents

$$V_a^\mu = \bar{q}\,\gamma^\mu \tfrac{1}{2}\lambda_a\, q\,,\;\; A_a^\mu = \bar{q}\,\gamma^\mu\gamma_5 \tfrac{1}{2}\lambda_a\, q\,,$$
$$S_a = \bar{q}\,\tfrac{1}{2}\lambda_a\, q\,,\qquad P_a = \bar{q}\, i\,\gamma_5\, \tfrac{1}{2}\lambda_a\, q$$

Include singlets, with $\lambda_0 = \sqrt{2/3} \times \mathbf{1}$, as well as

$$\omega = \frac{1}{16\pi^2}\,\mathrm{tr}_c\, G_{\mu\nu}\tilde{G}^{\mu\nu}$$

- Can collect all of the Green functions formed with these operators in a generating functional: Perturb the system with external fields $v_\mu^a(x)$, $a_\mu^a(x)$, $s_a(x)$, $p^a(x)$, $\theta(x)$
Replace the Lagrangian of the massless theory

$$\mathcal{L}_0 = -\frac{1}{2g^2}\,\mathrm{tr}_c\, G_{\mu\nu}G^{\mu\nu} + \bar{q}\, i\gamma^\mu(\partial_\mu - i\, G_\mu)\, q$$

by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_1 = v_\mu^a V_a^\mu + a_\mu^a A_a^\mu - s^a S_a - p^a P_a - \theta\,\omega$$

- Quark mass terms are included in the external field $s_a(x)$

- $|0\,\text{in}\rangle$: system is in ground state for $x^0 \to -\infty$
  Probability amplitude for finding ground state when $x^0 \to +\infty$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0\,\text{out}|0\,\text{in}\rangle_{v,a,s,p,\theta}$$

- Expressed in terms of ground state of $\mathcal{L}_0$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0|\,T\,\text{exp}\,i\!\int\! dx\mathcal{L}_1\,|0\rangle$$

- Expansion of $S_{\text{QCD}}\{v,a,s,p,\theta\}$ in powers of the external fields yields the connected parts of the Green functions of the massless theory

$$S_{\text{QCD}}\{v,a,s,p,\theta\} = -\int\! dx\, s_a(x)\langle 0|\,S^a(x)\,|0\rangle$$
$$+\tfrac{i}{2}\!\int\! dx dy\, a^a_\mu(x) a^b_\nu(y)\langle 0|\,T A^\mu_a(x) A^\nu_b(y)\,|0\rangle_{\text{conn}} + \ldots$$

- $S_{\text{QCD}}\{v,a,s,p,\theta\}$ is referred to as the *generating functional* of QCD

- For Green functions of full QCD, set

$$s_a(x) = m_a + \tilde{s}_a(x)\,, \qquad m_a = \text{tr}\lambda_a\, m$$

  and expand around $\tilde{s}_a(x) = 0$

- Path integral representation for generating functional:

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p\}} = \mathcal{N}\int [dG]\, e^{\,i\int dx\, \mathcal{L}_{\text{G}}}\ \det D}$$

$$\mathcal{L}_{\text{G}} = -\frac{1}{2g^2}\,\underset{c}{\text{tr}}\, G_{\mu\nu}G^{\mu\nu} - \frac{\theta}{16\pi^2}\,\underset{c}{\text{tr}}\, G_{\mu\nu}\tilde{G}^{\mu\nu}$$

$$D = i\gamma^{\mu}\{\partial_{\mu} - i(G_{\mu} + v_{\mu} + a_{\mu}\gamma_5)\} - s - i\gamma_5 p$$

G_{μ} is matrix in colour space

v_{μ}, a_{μ}, s, p are matrices in flavour space

$v_{\mu}(x) \equiv \frac{1}{2}\lambda_a\, v_{\mu}^a(x)$, etc.

8. Ward identities

Symmetry in terms of Green functions

- Lagrangian is invariant under

$$q_{\mathsf{R}}(x) \to V_{\mathsf{R}}(x)\, q_{\mathsf{R}}(x)\,, \quad q_{\mathsf{L}}(x) \to V_{\mathsf{L}}(x)\, q_{\mathsf{L}}(x)$$
$$V_{\mathsf{R}}(x), V_{\mathsf{L}}(x) \in \mathsf{U}(3)$$

provided the external fields are transformed with

$$v'_\mu + a'_\mu = V_{\mathsf{R}}(v_\mu + a_\mu)V_{\mathsf{R}}^\dagger - i\partial_\mu V_{\mathsf{R}} V_{\mathsf{R}}^\dagger$$
$$v'_\mu - a'_\mu = V_{\mathsf{L}}(v_\mu - a_\mu)V_{\mathsf{L}}^\dagger - i\partial_\mu V_{\mathsf{L}} V_{\mathsf{L}}^\dagger$$
$$s' + i\,p' = V_{\mathsf{R}}(s + i\,p)V_{\mathsf{L}}^\dagger$$

The operation takes the Dirac operator into

$$D' = \left\{P_- V_{\mathsf{R}} + P_+ V_{\mathsf{L}}\right\} D \left\{P_+ V_{\mathsf{R}}^\dagger + P_- V_{\mathsf{L}}^\dagger\right\}$$
$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

- $\det D$ requires regularization

$\nexists$ symmetric regularization

$\Rightarrow$ $\det D' \neq \det D$, only $|\det D'| = |\det D|$

symmetry does not survive quantization

- Change in det D can explicitly be calculated

  For an infinitesimal transformation

  $V_{\rm R} = \mathbf{1} + i\,\alpha + i\beta + \ldots\,, \qquad V_{\rm L} = \mathbf{1} + i\,\alpha - i\beta + \ldots$

  the change in the determinant is given by

  $$\det D' = \det D\; e^{-i\int dx\,\{2\langle\beta\rangle\omega + \langle\beta\Omega\rangle\}}$$

  $$\langle A\rangle \equiv \mathrm{tr}\,A$$

  $$\omega = \frac{1}{16\pi^2}\,\mathrm{tr}_c\, G_{\mu\nu}\tilde{G}^{\mu\nu} \qquad \text{gluons}$$

  $$\Omega = \frac{N_c}{4\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu v_\nu \partial_\rho v_\sigma + \ldots \qquad \text{ext. fields}$$

- Consequence for generating functional:
  The term with ω amounts to a change in θ,
  can be compensated by $\theta' = \theta - 2\,\langle\beta\rangle$
  Pull term with $\langle\beta\Omega\rangle$ outside the path integral

$$\Rightarrow \boxed{S_{\rm QCD}\{v', a', s', p', \theta'\} = S_{\rm QCD}\{v, a, s, p, \theta\} - \int dx \langle\beta\Omega\rangle}$$

$$\boxed{S_{\mathrm{QCD}}\{v',a',s',p',\theta'\} = S_{\mathrm{QCD}}\{v,a,s,p,\theta\} - \int dx \langle \beta\Omega \rangle}$$

- S_{QCD} is invariant under $\mathrm{U(3)_R \times U(3)_L}$, except for a specific change due to the anomalies
- Relation plays key role in low energy analysis: collects all of the Ward identities
  For the octet part of the axial current,e.g.

$$\partial^x_\mu \langle 0|\, T A^\mu_a(x)\, P_b(y)\, |0\rangle = -\tfrac{1}{4}\, i\, \delta(x-y) \langle 0|\, \overline{q}\{\lambda_a,\lambda_b\} q\, |0\rangle$$
$$+ \langle 0|\, T\overline{q}(x)\, i\gamma_5 \{m, \tfrac{1}{2}\lambda_a\} q(x)\, P_b(y)\, |0\rangle$$

- Symmetry of the generating functional implies the operator relations

$$\partial_\mu V^\mu_a = \overline{q}\, i[m, \tfrac{1}{2}\lambda_a] q\,, \qquad a = 0,\ldots,8$$
$$\partial_\mu A^\mu_a = \overline{q}\, i\gamma_5 \{m, \tfrac{1}{2}\lambda_a\} q\,, \qquad a = 1,\ldots,8$$
$$\partial_\mu A^\mu_0 = \sqrt{\tfrac{2}{3}}\, \overline{q}\, i\gamma_5 m q + \sqrt{6}\, \omega$$

- Textbook derivation of the Ward identities goes in inverse direction, but is slippery formal manipulations, anomalies ?

9. Low energy expansion

- If the spectrum has an energy gap

$\Rightarrow$ no singularities in scattering amplitudes or Green functions near $p = 0$

$\Rightarrow$ low energy behaviour may be analyzed with Taylor series expansion in powers of p

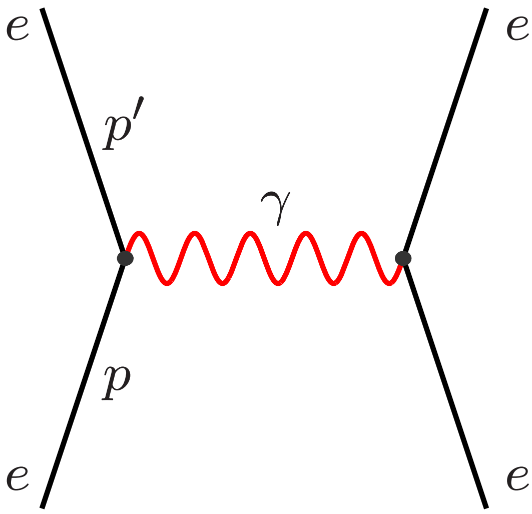
$$f(t) = 1 + \tfrac{1}{6}\langle r^2 \rangle\, t + \ldots \quad \text{form factor}$$
$$T(p) = a + b\, p^2 + \ldots \quad \text{scattering amplitude}$$

Cross section dominated by S–wave scattering length $\qquad \dfrac{d\sigma}{d\Omega} \simeq |a|^2$

- Expansion parameter: $\dfrac{p}{m} = \dfrac{\text{momentum}}{\text{energy gap}}$

- Taylor series only works if the spectrum has an energy gap, i.e. if there are <u>no massless particles</u>

- Illustration: Coulomb scattering



Photon exchange ⇒ pole at $t = 0$

$$T = \frac{e^2}{(p' - p)^2}$$

Scattering amplitude does not admit
Taylor series expansion in powers of p

- QCD does have an energy gap
but the gap is very small: M_π

⇒ Taylor series has very small radius of
convergence, useful only for $p < M_\pi$

- Massless QCD contains infrared singularities due to the Nambu-Goldstone bosons
- For $m_u = m_d = 0$, pion exchange gives rise to poles and branch points at $p = 0$

$\Rightarrow$ Low energy expansion is not a Taylor series, contains logarithms

Singularities due to Nambu-Goldstone bosons can be worked out with an effective field theory **Chiral Perturbation Theory**

Weinberg, Dashen, Pagels, Gasser, . . .

- Chiral perturbation theory correctly reproduces the infrared singularities of QCD
- Quantities of interest are expanded in powers of external momenta and quark masses
- Expansion has been worked out to next-to-leading order for many quantities "Chiral perturbation theory to one loop"
- In quite a few cases, the next-to-next-to-leading order is also known

- Properties of the Nambu-Goldstone bosons are governed by the hidden symmetry that is responsible for their occurrence
- Focus on the singularities due to the pions

$$H_{\mathrm{QCD}} = H_0 + H_1$$
$$H_1 = \int d^3x \, \{m_u \bar{u} u + m_d \bar{d} d\}$$

H_0 is invariant under $\mathrm{G} = \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}}$

$|0\rangle$ is invariant under $\mathrm{H} = \mathrm{SU}(2)_{\mathrm{V}}$

mass term of strange quark is included in H_0

- Treat H_1 as a perturbation

Expansion in powers of H_1	$\Longleftrightarrow$	Expansion in powers of m_u, m_d

- Extension to $\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$ straightforward: include singularities due to exchange of K, η

10. Effective Lagrangian

- Replace quarks and gluons by pions

$$\vec{\pi}(x) = \{\pi^1(x), \pi^2(x), \pi^3(x)\}$$
$$\mathcal{L}_{\text{QCD}} \;\rightarrow\; \mathcal{L}_{\textit{eff}}$$

- Central claim:

A. Effective theory yields alternative representation for generating functional of QCD

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{\textit{eff}} \int [d\pi] e^{i\int dx \mathcal{L}_{\textit{eff}}\{\vec{\pi},v,a,s,p,\theta\}}}$$

B. $\mathcal{L}_{\textit{eff}}$ has the same symmetries as $\mathcal{L}_{\text{QCD}}$

  - Lagrangian of QCD is invariant under

    $q_{\text{R}}(x) \rightarrow V_{\text{R}}(x)\, q_{\text{R}}(x)\,, \quad q_{\text{L}}(x) \rightarrow V_{\text{L}}(x)\, q_{\text{L}}(x)$

    $V_{\text{R}}(x), V_{\text{L}}(x) \in \text{U}(3)$

provided the external fields are transformed with

$$v'_\mu + a'_\mu = V_{\text{R}}(v_\mu + a_\mu)V_{\text{R}}^\dagger - i\partial_\mu V_{\text{R}} V_{\text{R}}^\dagger$$
$$v'_\mu - a'_\mu = V_{\text{L}}(v_\mu - a_\mu)V_{\text{L}}^\dagger - i\partial_\mu V_{\text{L}} V_{\text{L}}^\dagger$$
$$s' + i\,p' = V_{\text{R}}(s + i\,p)V_{\text{L}}^\dagger$$

  - $S_{\text{QCD}}\{v, a, s, p, \theta\}$ invariant modulo anomalies

- Action of the symmetry on the meson field:

$$U' = V_{\mathsf{R}} \cdot U \cdot V_{\mathsf{L}}^{\dagger}$$

- $\mathcal{L}_{\mathit{eff}}$ also invariant modulo anomalies:

$$\boxed{\mathcal{L}_{\mathit{eff}}\{U', v', a', s', p', \theta'\} = \mathcal{L}_{\mathit{eff}}\{U, v, a, s, p, \theta\}} \qquad (\star)$$

11. Explicit construction of $\mathcal{L}_{\mathit{eff}}$

Construct the general solution of $(\star)$

- First ignore the external fields,

$$\mathcal{L}_{\mathit{eff}} = \mathcal{L}_{\mathit{eff}}(U, \partial U, \partial^2 U, \ldots)$$

Order in the number of derivatives

- Symmetry fixes leading term up to a constant:

$$\boxed{\mathcal{L}_{\mathit{eff}} = \frac{F^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)}$$

$$\boxed{\mathcal{L}_{eff} = \frac{F^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)}$$

- Lagrangian of the nonlinear σ-model
- Expansion in powers of $\vec{\pi}$:

$$U = \exp i\,\vec{\pi}\cdot\vec{\tau} = \mathbf{1} + i\,\vec{\pi}\cdot\vec{\tau} - \tfrac{1}{2}\,\vec{\pi}^{\,2} + \ldots$$

$\Rightarrow$ $\mathcal{L}_{eff} = \frac{F^2}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{F^2}{48}\mathrm{tr}\{[\partial_\mu\pi,\pi]\,[\partial^\mu\pi,\pi]\} + \ldots$

For the kinetic term to have the standard normalization: rescale the pion field, $\vec{\pi} \to \vec{\pi}/F$

$$\mathcal{L}_{eff} = \tfrac{1}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{1}{48F^2}\mathrm{tr}\{[\partial_\mu\pi,\pi]\,[\partial^\mu\pi,\pi]\} + \ldots$$

$\Rightarrow$ a. Symmetry requires the pions to interact

b. Derivative coupling: Nambu-Goldstone bosons only interact if their momentum does not vanish $\Rightarrow$ $\not{\lambda}\pi^4$

- Expression given for $\mathcal{L}_{eff}$ only holds if the external fields are turned off. Also, tr$(\partial_\mu U \partial^\mu U^\dagger)$ is invariant only under global transformations

  Suffices to replace $\partial_\mu U$ by

  $$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu)$$

  In contrast to tr$(\partial_\mu U \partial^\mu U^\dagger)$, the term tr$(D_\mu U D^\mu U^\dagger)$ is invariant under local SU(2)$_\text{R}\times$ SU(2)$_\text{L}$

- Can construct further invariants: $s + ip$ transforms like $U \Rightarrow$ tr$\{(s+ip)U^\dagger\}$ is invariant Violates parity, but tr$\{(s+ip)U^\dagger\}+$tr$\{(s-ip)U\}$ is even under $p \to -p, \vec{\pi} \to -\vec{\pi}$

  In addition, $\exists$ invariant independent of U: $D_\mu\theta D^\mu\theta$, with $D_\mu\theta = \partial_\mu\theta + 2\,\text{tr}(a_\mu)$

- Count the external fields as
  $\theta = O(1), \quad v_\mu, a_\mu = O(p), \quad s, p = O(p^2)$

- Derivative expansion yields string of the form

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots$$

- Full expression for leading term:

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle + h_0 D_\mu \theta D^\mu \theta}$$

$$\chi \equiv 2\,B\,(s + ip)\,, \qquad \langle X \rangle \equiv \mathrm{tr}(X)$$

- Contains 3 constants: F, B, h_0
  "effective coupling constants"
  "low energy constants", LEC

- Next-to-leading order:

$$\mathcal{L}^{(4)} = \frac{\ell_1}{4}\langle D_\mu U D^\mu U \rangle^2 + \frac{\ell_2}{4}\langle D_\mu U D_\nu U \rangle \langle D^\mu U D^\nu U \rangle + \frac{\ell_3}{4}\langle \chi U^\dagger + U \chi^\dagger \rangle^2 + \frac{\ell_4}{4}\langle D_\mu \chi D^\mu U^\dagger + D_\mu U D^\mu \chi^\dagger \rangle + \dots$$

- Number of effective coupling constants rapidly grows with the order of the expansion

- Infinitely many effective coupling constants
  Symmetry does not determine these
  Predictivity ?
- Essential point: If $\mathcal{L}_{eff}$ is known to given order $\Rightarrow$ can work out low energy expansion of the Green functions to that order (Weinberg 1979)
- NLO expressions for F_π, M_π involve 2 new coupling constants: ℓ_3, ℓ_4.

  In the $\pi\pi$ scattering amplitude, two further coupling constants enter at NLO: ℓ_1, ℓ_2.
- Note: effective theory is a quantum field theory
  Need to perform the path integral

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{eff} \int [d\pi] e^{i\int dx \mathcal{L}_{eff}\{\vec{\pi},v,a,s,p,\theta\}}}$$

- Classical theory $\Leftrightarrow$ tree graphs
  Need to include graphs with loops
- Power counting in dimensional regularization:
  Graphs with ℓ loops are suppressed by factor $p^{2\ell}$ as compared to tree graphs

$\Rightarrow$ Leading contributions given by tree graphs
Graphs with one loop contribute at next-to-leading order, etc.

- The leading contribution to S_{QCD} is given by the sum of all tree graphs = classical action:

$$S_{\text{QCD}}\{v,a,s,p,\theta\} = \underset{U(x)}{\text{extremum}} \int dx\, \mathcal{L}_{\textit{eff}}\{U,v,a,s,p,\theta\}$$

III. Illustrations

12. Some tree level calculations

A. Extracting the quark condensate from the generating functional

- To calculate the quark condensate of the massless theory, it suffices to consider the generating functional for $v = a = p = \theta = 0$ and to take a constant scalar external field

$$s = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

- Expansion in powers of m_u and m_d treats

$$H_1 = \int d^3x \, \{m_u \bar{u}u + m_d \bar{d}d\} \text{ as a perturbation}$$

$$S_{\text{QCD}}\{0, 0, m, 0, 0\} = S^0_{\text{QCD}} + S^1_{\text{QCD}} + \ldots$$

- S^0_{QCD} is independent of the quark masses (cosmological constant)
- S^1_{QCD} is linear in the quark masses

- First order in m_u, m_d $\Rightarrow$ expectation value of H_1 in unperturbed ground state is relevant

$$S^1_{\text{QCD}} = -\int dx \langle 0|\, m_u \bar{u}u + m_d \bar{d}d \,|0\rangle$$

$\Rightarrow$ $\langle 0|\bar{u}u\,|0\rangle$ and $\langle 0|\bar{d}d\,|0\rangle$ are the coefficients of the terms in S_{QCD} that are linear in m_u and m_d

B. Condensate in terms of effective theory

- Need the effective action for $v = a = p = \theta = 0$ to first order in s

$\Rightarrow$ classical level of effective theory suffices.

- extremum of the classical action: $U = \mathbf{1}$

$$S^1_{\text{QCD}} = \int dx F^2 B(m_u + m_d)$$

- comparison with

$$S^1_{\text{QCD}} = -\int dx \langle 0|\, m_u \bar{u}u + m_d \bar{d}d \,|0\rangle \quad \text{yields}$$

$$\boxed{\langle 0|\bar{u}u\,|0\rangle = \langle 0|\bar{d}d\,|0\rangle = -F^2 B} \qquad (1)$$

C. Evaluation of M_π at tree level

- In classical theory, the square of the mass is the coefficient of the term in the Lagrangian that is quadratic in the meson field:

$$\frac{F^2}{4}\langle \chi U^\dagger + U\chi^\dagger\rangle = \frac{F^2 B}{2}\langle m(U^\dagger + U)\rangle$$
$$= F^2 B(m_u + m_d)\{1 - \frac{\vec{\pi}^{\,2}}{2F^2} + \ldots\}$$

Hence $\boxed{M_\pi^2 = (m_u + m_d)B}$ (2)

- Tree level result for F_π:

$$\boxed{F_\pi = F} \qquad (3)$$

- (1) + (2) + (3) $\Rightarrow$ GMOR relation:

$$\boxed{M_\pi^2 = \frac{(m_u + m_d)\,|\langle 0|\bar{u}u\,|0\rangle|}{F_\pi^2}}$$

13. M_π beyond tree level

- The formula $M_\pi^2 = (m_u + m_d)B$ only holds at tree level, represents leading term in expansion of M_π^2 in powers of m_u, m_d
- Disregard isospin breaking: set $m_u = m_d = m$

A. M_π to 1 loop

- Claim: at next-to-leading order, the expansion of M_π^2 in powers of m contains a logarithm:

$$M_\pi^2 = M^2 - \frac{1}{2}\frac{M^4}{(4\pi F)^2}\ln\frac{\Lambda_3^2}{M^2} + O(M^6)$$
$$M^2 \equiv 2mB$$

- Proof: Pion mass $\Leftrightarrow$ pole position, for instance in the Fourier transform of $\langle 0|\, T A_a^\mu(x) A_b^\nu(y)\, |0\rangle$

  Suffices to work out the perturbation series for this object to one loop of the effective theory

- Result (exercise # 5):

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\,\frac{1}{i}\Delta(0,M^2) + O(M^6)$$

$\Delta(0,M^2)$ is the propagator at the origin (exercise # 2):

$$\Delta(0,M^2) = \frac{1}{(2\pi)^d}\int\frac{d^dp}{M^2-p^2-i\epsilon}$$
$$= i\,(4\pi)^{-d/2}\,\Gamma(1-d/2)M^{d-2}$$

- Contains a pole at $d=4$:

$$\Gamma\,(1-d/2) = \frac{2}{d-4} + \ldots$$

- Divergent part is proportional to M^2:

$$M^{d-2} = M^2\mu^{d-4}(M/\mu)^{d-4} = M^2\mu^{d-4}e^{(d-4)\,\ln(M/\mu)}$$
$$= M^2\mu^{d-4}\{1+(d-4)\ln(M/\mu)+\ldots\}$$

- Denote the singular factor by

$$\lambda \equiv \frac{1}{2}(4\pi)^{-d/2}\,\Gamma(1-d/2)\,\mu^{d-4}$$
$$= \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}(\ln 4\pi + \Gamma'(1) + 1) + O(d-4)\right\}$$

- The propagator at the origin then becomes

$$\frac{1}{i}\Delta(0,M^2)=M^2\left\{2\lambda+\frac{1}{16\pi^2}\ln\frac{M^2}{\mu^2}+O(d-4)\right\}$$

- In the expression for M_π^2

$$M_\pi^2 = M^2+\frac{2\,\ell_3 M^4}{F^2}+\frac{M^2}{2F^2}\frac{1}{i}\Delta(0,M^2)+O(M^6)$$

the divergence can be absorbed in ℓ_3:

$$\ell_3 = -\frac{1}{2}\lambda+\ell_3^{\text{ren}}$$

- ℓ_3^{ren} depends on the renormalization scale μ

$$\ell_3^{\text{ren}} = \frac{1}{64\pi^2}\ln\frac{\mu^2}{\Lambda_3^2} \quad \text{running coupling constant}$$

- Λ_3 is the ren. group invariant scale of ℓ_3

Net result for M_π^2

$$\boxed{M_\pi^2 = M^2-\frac{1}{2}\frac{M^4}{(4\pi F)^2}\ln\frac{\Lambda_3^2}{M^2}+O(M^6)}$$

$\Rightarrow$ M_π^2 contains a chiral logarithm at NLO

- Crude estimate for Λ_3, based on SU(3) mass formulae for the pseudoscalar octet:

$$0.2 \text{ GeV} < \Lambda_3 < 2 \text{ GeV}$$

$$\bar{\ell}_3 \equiv \ln \frac{\Lambda_3^2}{M_\pi^2} = 2.9 \pm 2.4$$ Gasser, L. 1984

$\exists$ better determination $\bar{\ell}_3$ on the lattice, to be discussed later

$\Rightarrow$ Next–to–leading term is small correction:

$$0.005 < \frac{1}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln \frac{\Lambda_3^2}{M_\pi^2} < 0.04$$

- Scale of the expansion is set by size of pion mass in units of decay constant:

$$\frac{M^2}{(4\pi F)^2} \simeq \frac{M_\pi^2}{(4\pi F_\pi)^2} = 0.0144$$

B. M_π to 2 loops

- Terms of order m_{quark}^3:

$$M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\,\ln\frac{\Lambda_3^2}{M^2} + \frac{17}{18}\,\frac{M^6}{(4\pi F)^4}\left(\ln\frac{\Lambda_M^2}{M^2}\right)^2 + k_M M^6 + O(M^8)$$

F is pion decay constant for $m_u = m_d = 0$
ChPT to two loops Colangelo 1995

- Coefficients $\frac{1}{2}$ and $\frac{17}{18}$ determined by symmetry

- Λ_3, Λ_M and $k_M \Longleftrightarrow$ coupling constants in $\mathcal{L}_{\mathit{eff}}$

14. F_π to one loop

- Also contains a logarithm at NLO:

$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\Lambda_4^2} + O(M^4)\right\}$$
$$M_\pi^2 = M^2\left\{1 + \frac{M^2}{32\pi^2F^2}\ln\frac{M^2}{\Lambda_3^2} + O(M^4)\right\}$$

F is pion decay constant in limit $m_u, m_d \to 0$

- Structure is the same, coefficients and scale of logarithm are different

15. Pion form factors

- Scalar form factor of the pion:

$$F_s(t) = \langle \pi(p')|\,\bar{q}\,q\,|\pi(p)\rangle \;, \qquad t = (p'-p)^2$$

- Definition of scalar radius:

$$F_s(t) = F_s(0)\left\{1 + \frac{1}{6}\langle r^2\rangle_s\, t + O(t^2)\right\}$$

- Low energy theorem:

$$\langle r^2\rangle_s = \frac{6}{(4\pi F)^2}\left\{\ln\frac{\Lambda_4^2}{M^2} - \frac{13}{12} + O(M^2)\right\}$$

⇒ In massless QCD, the scalar radius diverges, because the density of the pion cloud only decreases with a power of the distance

- Same infrared singularity also occurs in the charge radius (e.m. current), but coefficient of the chiral logarithm is 6 times smaller:

$$\langle r^2\rangle_s = \frac{6}{(4\pi F)^2}\left\{\ln\frac{\Lambda_4^2}{M^2} - \frac{13}{12} + O(M^2)\right\}$$

$$\langle r^2\rangle_{em} = \frac{1}{(4\pi F)^2}\left\{\ln\frac{\Lambda_6^2}{M^2} - 1 + O(M^2)\right\}$$

⇒ $\langle r^2\rangle_s > \langle r^2\rangle_{em}$ if M small enough

- $\langle r^2 \rangle_{em}$ can be determined experimentally

$$\langle r^2 \rangle_{em} = 0.439 \pm 0.008\,\text{fm}^2$$

NA7 Collaboration, NP B277 (1986) 168

- Scalar form factor of the pion can be calculated by means of dispersion theory
- Result for the slope:

$$\langle r^2 \rangle_s = 0.61 \pm 0.04\,\text{fm}^2$$

Colangelo, Gasser, L., Nucl. Phys. 2001

⇒ Corresponding value of the scale Λ_4:

$$\Lambda_4 = 1.26 \pm 0.14\,\text{GeV}$$

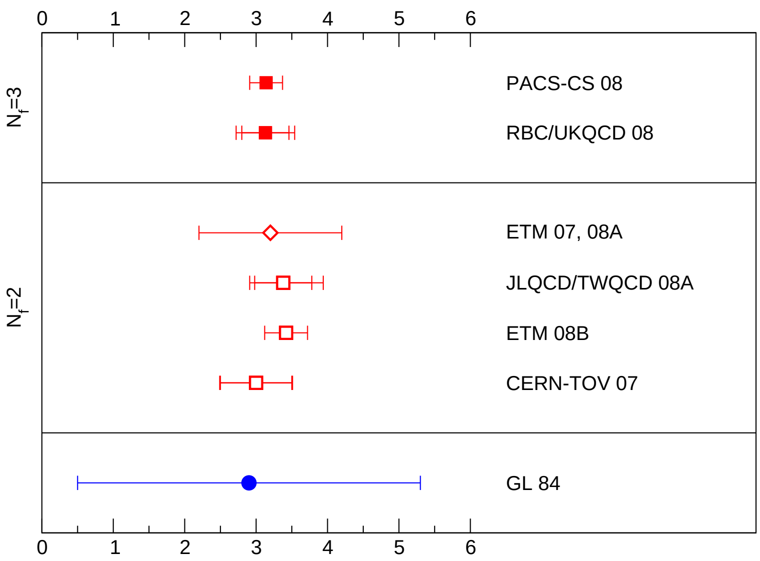
16. Lattice results for M_π, F_π

A. Results for M_π

- Determine the scale Λ_3 by comparing the lattice results for M_π as function of m with the χPT formula

$$M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\,\ln\frac{\Lambda_3^2}{M^2} + O(M^6)$$

$$M^2 \equiv 2Bm$$



lattice results for $\bar{\ell}_3$

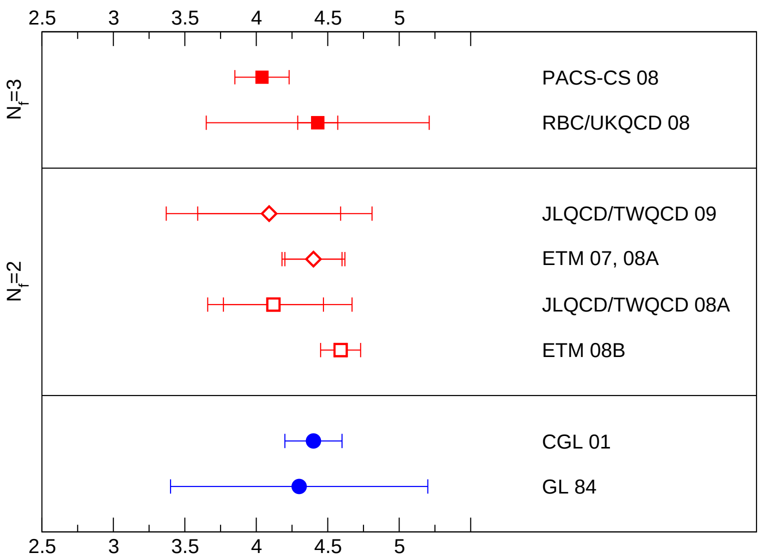
Horizontal axis shows the value of $\bar{\ell}_3 \equiv \ln\dfrac{\Lambda_3^2}{M_\pi^2}$

Range for Λ_3 obtained in 1984 corresponds to $\bar{\ell}_3 = 2.9 \pm 2.4$

Result of RBC/UKQCD 2008: $\bar{\ell}_3 = 3.13 \pm \underset{stat}{0.33} \pm \underset{syst}{0.24}$

B. Results for F_π

$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\Lambda_4^2} + O(M^4)\right\}$$



Horizontal axis shows the value of $\bar{\ell}_4 \equiv \ln\frac{\Lambda_4^2}{M_\pi^2}$

- Lattice results beautifully confirm the prediction for the sensitivity of F_π to m_u, m_d:

$$\frac{F_\pi}{F} = 1.072 \pm 0.007$$ Colangelo, Dürr 2004

17. $\pi\pi$ scattering

A. Low energy scattering of pions

- Consider scattering of pions with $\vec{p} = 0$
- At $\vec{p} = 0$, only the S-waves survive (angular momentum barrier). Moreover, these reduce to the scattering lengths
- Bose statistics: S-waves cannot have $I = 1$, either have $I = 0$ or $I = 2$

⇒ At $\vec{p} = 0$, the $\pi\pi$ scattering amplitude is characterized by two constants: a_0^0, a_0^2

- Chiral symmetry suppresses the interaction at low energy: Nambu-Goldstone bosons of zero momentum do not interact

⇒ a_0^0, a_0^2 disappear in the limit $m_u, m_d \to 0$

⇒ $a_0^0, a_0^2 \sim M_\pi^2$ measure symmetry breaking

B. Tree level of χPT

- Low Energy theorem Weinberg 1966:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} + O(M_\pi^4)$$
$$a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} + O(M_\pi^4)$$

$\Rightarrow$ Chiral symmetry predicts a_0^0, a_0^2 in terms of F_π

- Accuracy is limited: Low energy theorem only specifies the first term in the expansion in powers of the quark masses
  Corrections from higher orders ?

C. Scattering lengths at 1 loop

- Next term in the chiral perturbation series:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left\{1 + \frac{9}{2}\,\frac{M_\pi^2}{(4\pi F_\pi)^2}\,\ln\frac{\Lambda_0^2}{M_\pi^2} + O(M_\pi^4)\right\}$$

- Coefficient of chiral logarithm unusually large
  Strong, attractive final state interaction

- Scale Λ_0 is determined by the coupling constants of $\mathcal{L}_{eff}^{(4)}$:

$$\frac{9}{2}\ln\frac{\Lambda_0^2}{M_\pi^2} = \frac{20}{21}\,\bar{\ell}_1 + \frac{40}{21}\,\bar{\ell}_2 - \frac{5}{14}\,\bar{\ell}_3 + 2\,\bar{\ell}_4 + \frac{5}{2}$$

- Information about $\bar{\ell}_1, \ldots, \bar{\ell}_4$?

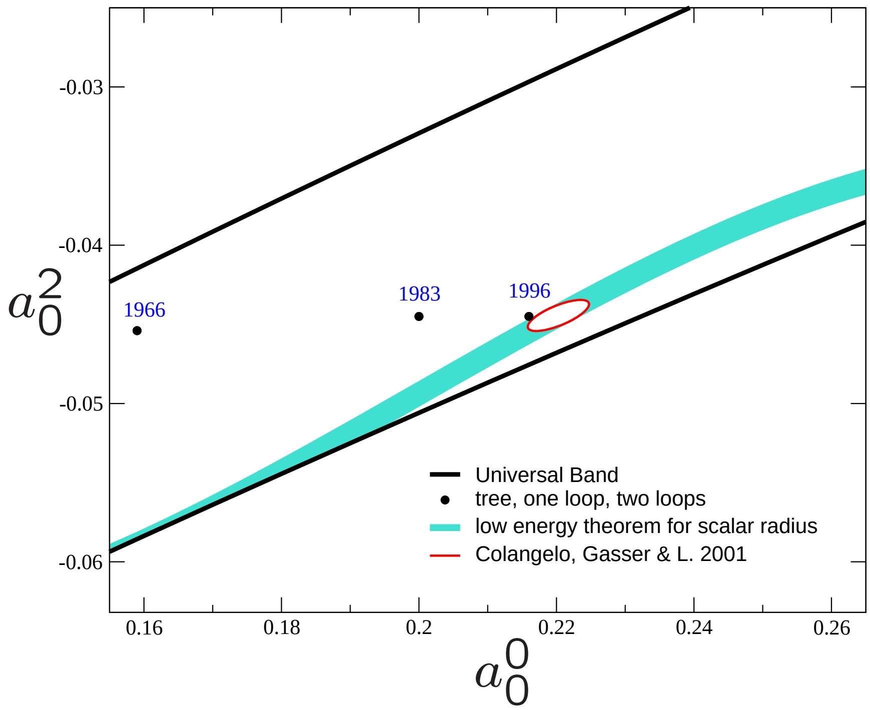
$\bar{\ell}_1, \bar{\ell}_2$	$\Longleftrightarrow$	momentum dependence of scattering amplitude

$\Rightarrow$ Can be determined phenomenologically

$\bar{\ell}_3, \bar{\ell}_4$	$\Longleftrightarrow$	dependence of scattering amplitude on quark masses

Have discussed their values already

D. Numerical predictions from χPT



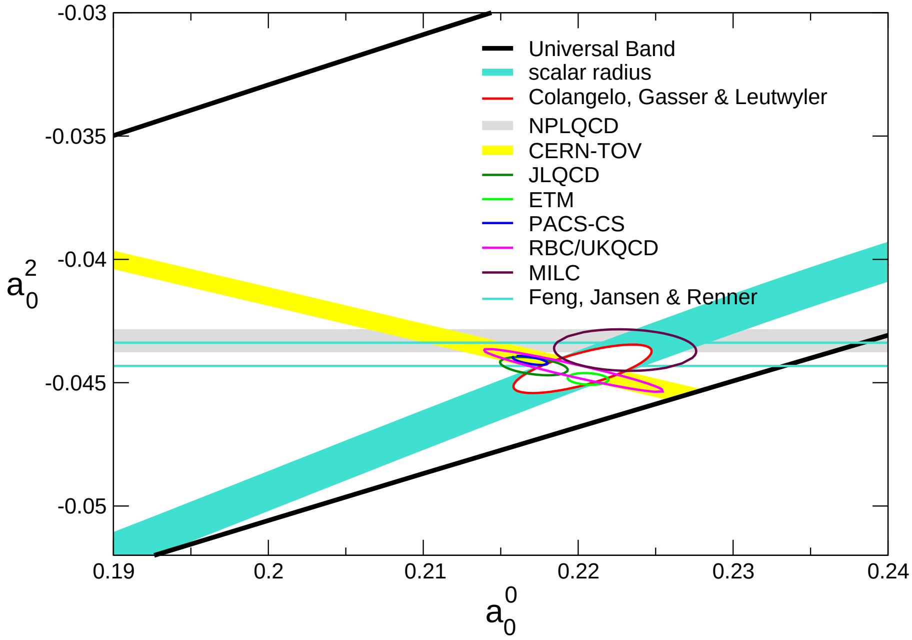
Sizable corrections in a_0^0

a_0^2 nearly stays put

E. Consequence of lattice results for ℓ_3, ℓ_4

- Uncertainty in prediction for a_0^0, a_0^2 is dominated by the uncertainty in the effective coupling constants ℓ_3, ℓ_4

- Can make use of the lattice results for these



F. Experiments concerning a_0^0, a_0^2

- Production experiments $\pi N \to \pi\pi N$,
  $\psi \to \pi\pi\omega$, $B \to D\pi\pi$, . . .

  Problem: pions are not produced in vacuo

⇒ Extraction of $\pi\pi$ scattering amplitude is not simple

  Accuracy rather limited

- $K^\pm \to \pi^+\pi^- e^\pm \nu$ data:
  CERN-Saclay, E865, NA48/2

- $K^\pm \to \pi^0\pi^0\pi^\pm$, $K^0 \to \pi^0\pi^0\pi^0$: cusp near threshold, NA48/2

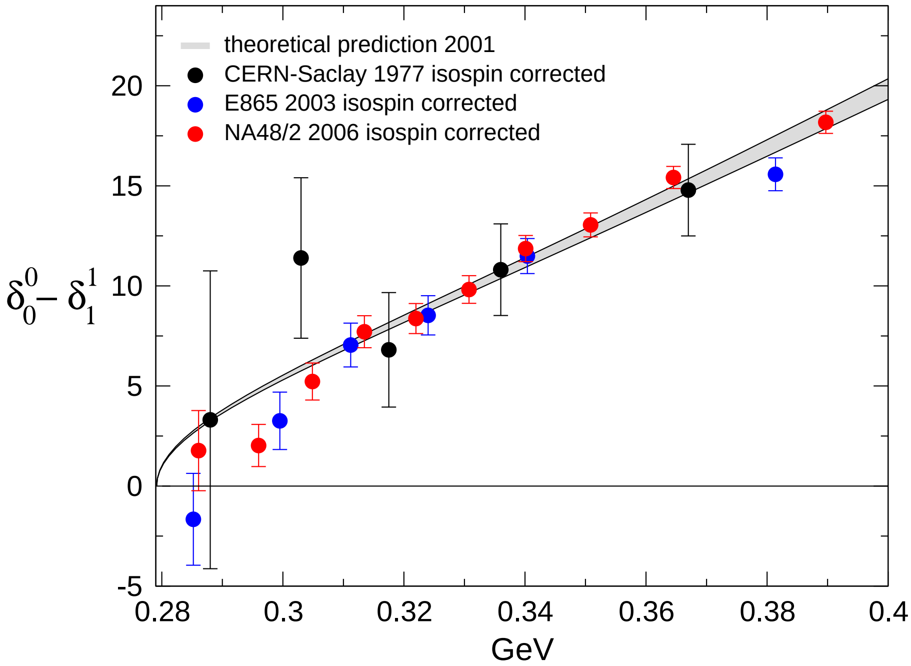
- $\pi^+\pi^-$ atoms, DIRAC

G. Results from K_{e4} decay

$K^{\pm} \to \pi^+ \pi^- e^{\pm} \nu$

- Allows clean measurement of $\delta_0^0 - \delta_1^1$

Theory predicts $\delta_0^0 - \delta_1^1$ as function of energy



Prediction: $a_0^0 = 0.220 \pm 0.005$

NA48/2: $a_0^0 = 0.2206 \pm \underset{stat}{0.0049} \pm \underset{syst}{0.0018} \pm \underset{theo}{0.0064}$

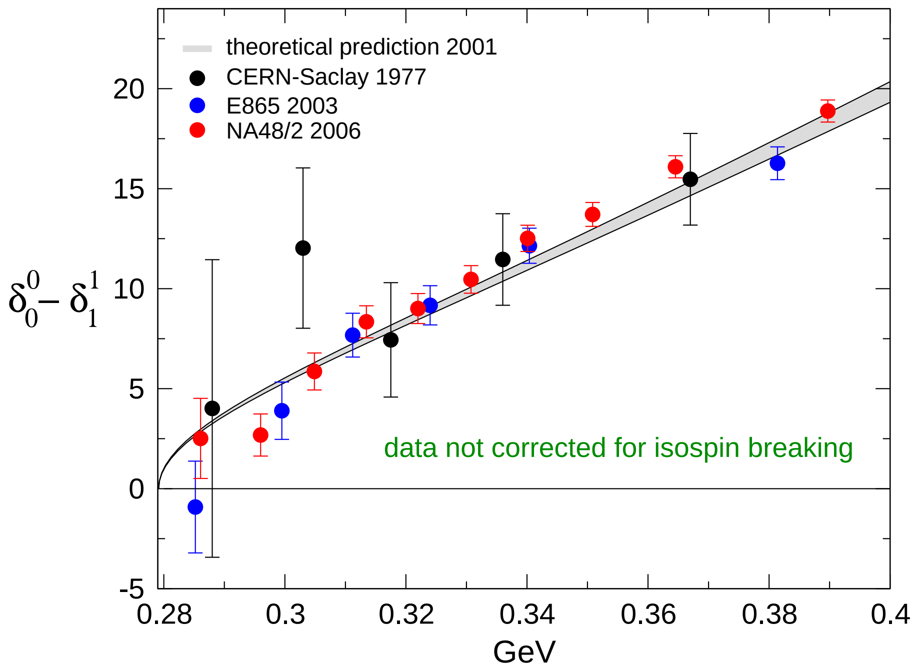
Bloch-Devaux, Chiral Dynamics 2009

- There was a discrepancy here, because a pronounced isospin breaking effect from

$$K \to \pi^0 \pi^0 e\nu \to \pi^+ \pi^- e\nu$$

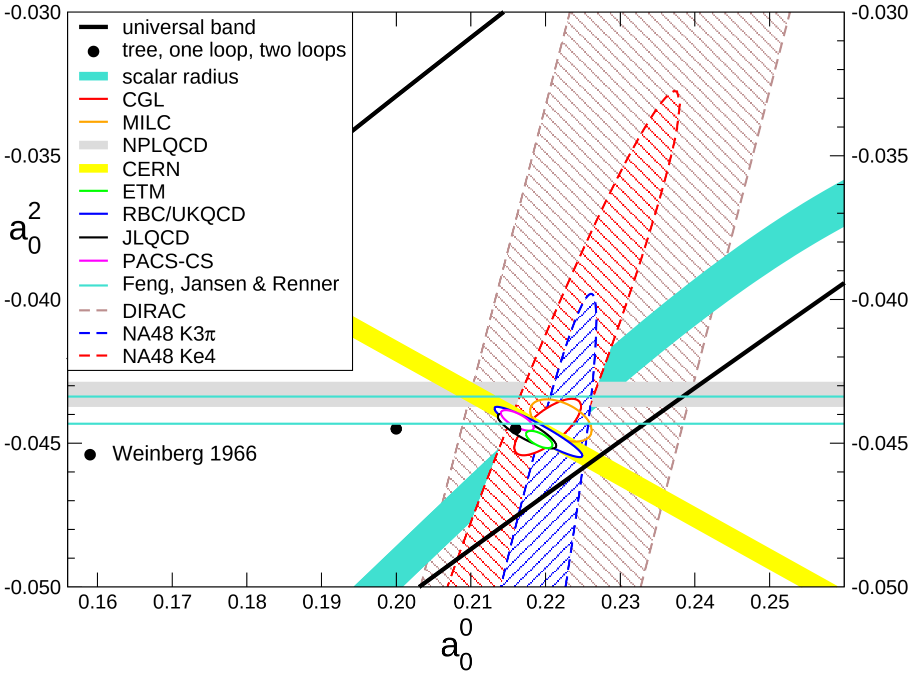
had not been accounted for in the data analysis

Colangelo, Gasser, Rusetsky 2007, Bloch-Devaux 2007



- The correction is not enormous, but matters: If a_0^0 is determined from the uncorrected NA48 data, the central value comes out higher than the theoretical prediction by about 4 times the uncertainty attached to this prediction.

H. Summary for a_0^0, a_0^2



18. Conclusions for SU(2)×SU(2)

- Expansion in powers of m_u, m_d yields a very accurate low energy representation of QCD
- Lattice results confirm the GMOR relation

⇒ M_π is dominated by the contribution from the quark condensate

⇒ Energy gap of QCD is understood very well

- Lattice approach allows an accurate measurement of the effective coupling constant ℓ_3 already now
- Even for ℓ_4, the lattice starts becoming competitive with dispersion theory

19. Expansion in powers of m_s

- Theoretical reasoning
  - The eightfold way is an approximate symmetry
  - The only (?) way to understand this within QCD: $m_s - m_d$, $m_d - m_u$ are small, can be treated as perturbations
  - Since $m_u, m_d \ll m_s$

  $\Rightarrow$ m_s can be treated as a perturbation

  $\Rightarrow$ Expect expansion in powers of m_s to work, but convergence to be comparatively slow

- This can now also be checked on the lattice

- Consider the limit $m_u, m_d \to 0$, m_s physical
    - F is value of F_π in this limit
    - B is value of $M_\pi^2/(m_u+m_d)$ in this limit
    - Σ is value of $|\langle 0|\,\bar{u}u\,|0\rangle|$ in this limit

- Exact relation: $\Sigma = F^2 B$

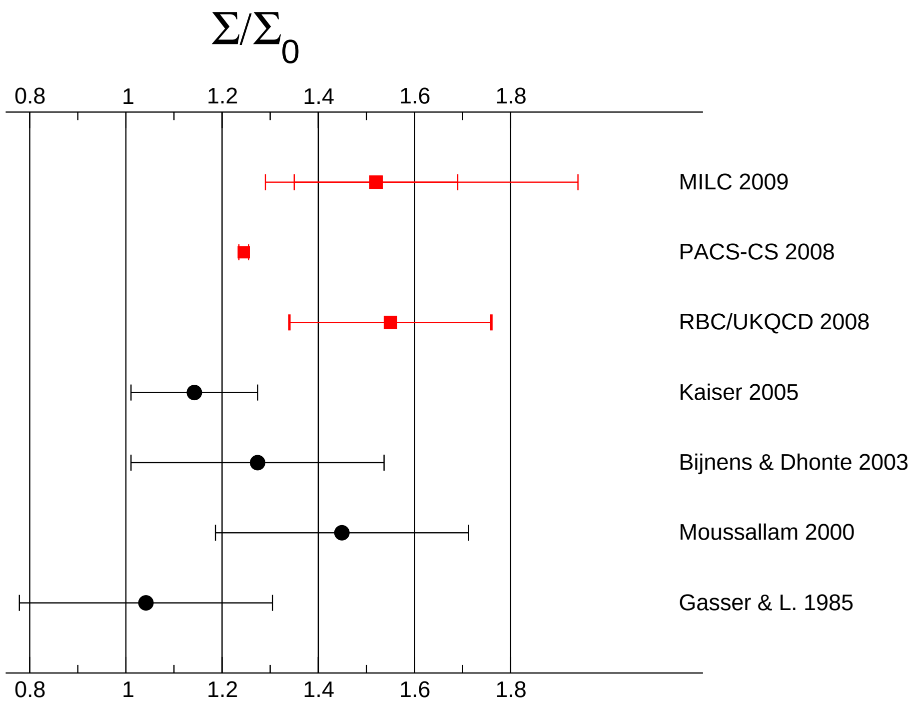
- F_0, B_0, Σ_0: values for $m_u = m_d = m_s = 0$

- $N_c \to \infty$: F, B, Σ become independent of m_s

$$F/F_0 \to 1,\ B/B_0 \to 1,\ \Sigma/\Sigma_0 \to 1$$

⇒ The differences $F/F_0 - 1$, $B/B_0 - 1$, $\Sigma/\Sigma_0 - 1$ measure the violations of the OZI rule

A. Condensate

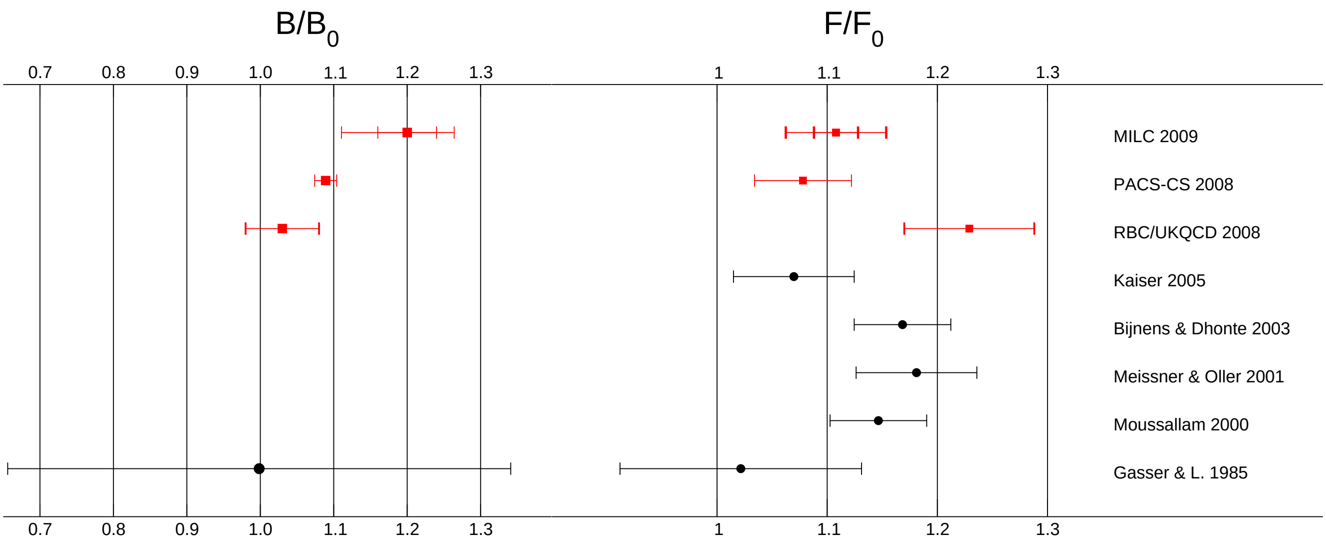


$$\Sigma = -\langle 0|\bar{u}u\,|0\rangle\Big|_{m_u,m_d\to 0}\ , \qquad \Sigma_0 = \Sigma\Big|_{m_s\to 0}$$

- PACS-CS indicates only modest OZI-violations
- MILC and RBC/UKQCD allow juicy violations

$\Rightarrow$ The lattice results do not yet allow to draw conclusions about the size of the OZI-violations in the quark condensate

B. Results for B, F



- F is the crucial factor in $\Sigma = F^2 B$
- Picture for size of OZI-violations in B, F remains unclear
- Main problem: systematic uncertainties of the lattice calculations

- If the central value $F/F_0 = 1.23$ of RBC/UKQCD were confirmed within small uncertainties, we would be faced with a qualitative puzzle:
  - F_π is the pion wave function at the origin
  - F_K is larger because one of the two valence quarks is heavier $\rightarrow$ moves more slowly $\rightarrow$ wave function more narrow $\rightarrow$ higher at the origin: $F_K/F_\pi \simeq 1.19$
  - $F/F_0 = 1.23$ indicates that the wave function is more sensitive to the mass of the sea quarks than to the mass of the valence quarks . . . very strange $\rightarrow$ most interesting if true
- No such puzzle with the PACS-CS results

C. Expansion to NLO

Involves the effective coupling constants L_4 and L_6 of the SU(3)×SU(3) Lagrangian:

$$F/F_0 = 1 + \frac{8\,\overline{M}_K^2}{F_0^2}\,L_4 + \chi\text{log} + \ldots$$

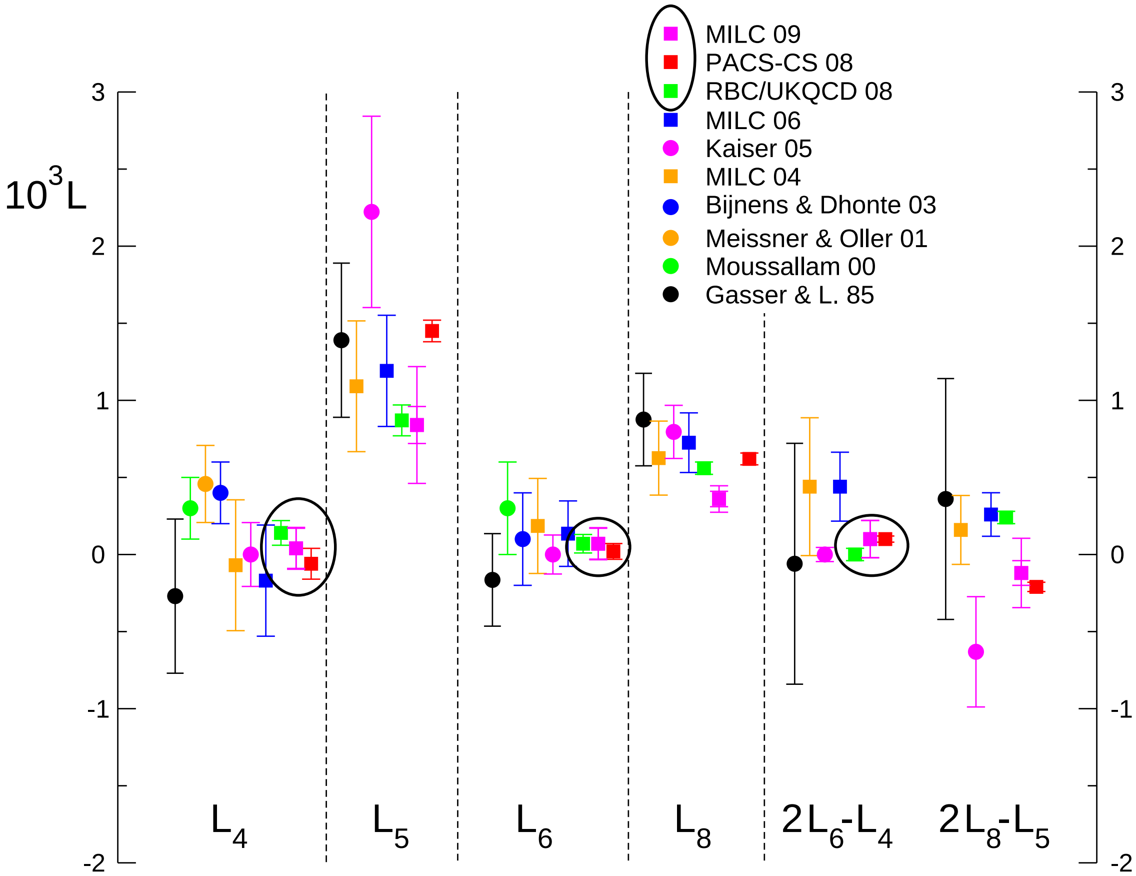
$$\Sigma/\Sigma_0 = 1 + \frac{32\,\overline{M}_K^2}{F_0^2}\,L_6 + \chi\text{log} + \ldots$$

$$B/B_0 = 1 + \frac{16\,\overline{M}_K^2}{F_0^2}\,(2L_6 - L_4) + \chi\text{log} + \ldots$$

$\overline{M}_K$ is the kaon mass for $m_u = m_d = 0$.

⇒ The LECs L_4 and L_6 measure the deviations from the OZI-rule

D. Effective coupling constants L_4, L_5, L_6, L_8



Numerical values shown refer to running scale $\mu = M_\rho$

⇒ For PACS-CS, only the statistical errors are indicated

- Latest lattice results for the OZI-violating coupling constants L_4 and L_6 are consistent with one another
- Indicate that the OZI-rule is well obeyed: values are close to zero
- For L_5 and L_8, the lattice results are less clear

20. Conclusions for SU(3)×SU(3)

- The crude estimates given 25 years ago for the LECs relevant at NLO are confirmed

⇒ Expansion in powers of m_s appears to work:

In all cases I know, where the calculation is under control, the truncation at low order yields a decent approximation

⇒ The picture looks coherent, also for SU(3)×SU(3)

- $m_s \gg m_u, m_d \Rightarrow$ higher orders more important
- For many observables ∃ representation to NNLO

Bijnens and collaborators

- Main problem: new LECs relevant at NNLO

  ∃ estimates based on resonance models

  Vector meson dominance √

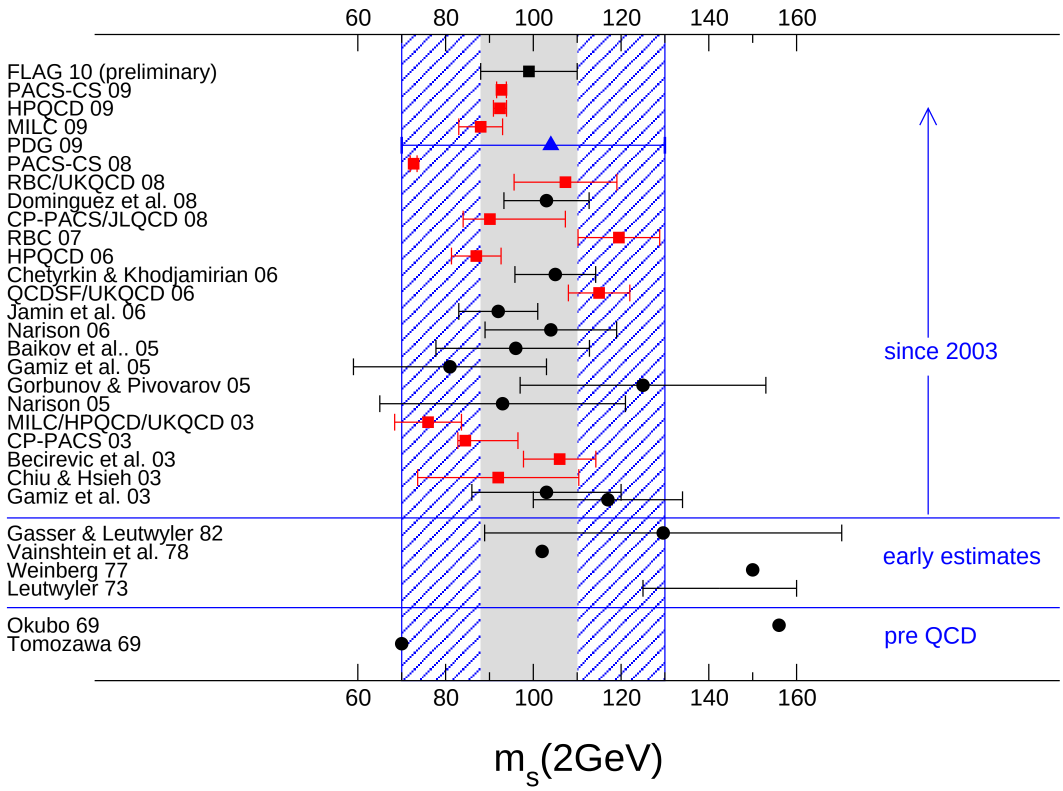
  Scalar meson dominance ?

  Dependence on m_u, m_d, m_s: scalar resonances
- Lattice results now start providing more precise values for the LECs, but the settling of dust is a slow process . . .

IV. Some recent results

21. Masses of the light quarks

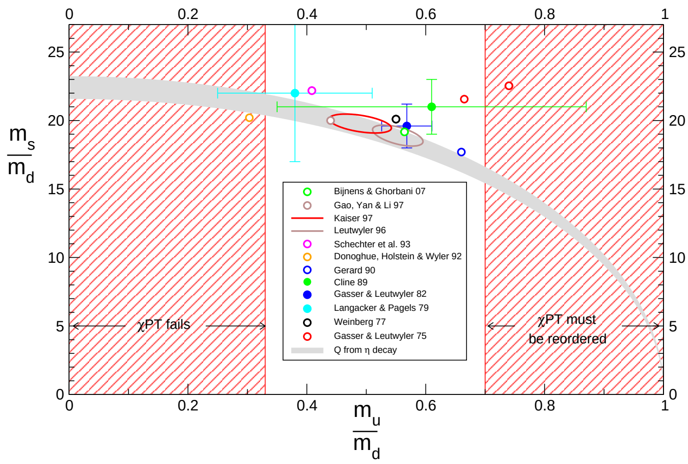
- χPT plays an important role in the analysis of lattice data: describes the dependence of the various observables on the quark masses and on the size of the box in terms of a few LECs



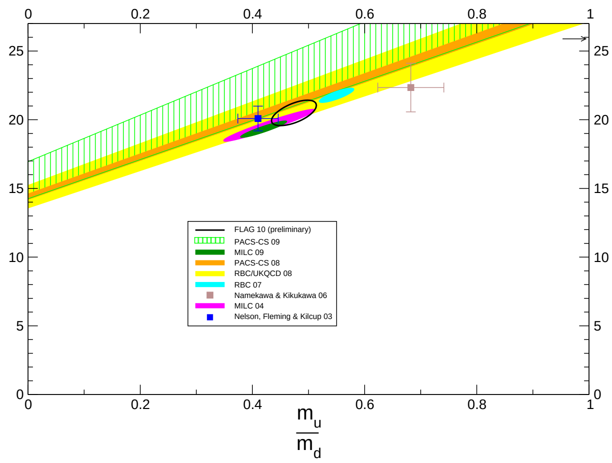
$$m_s(2\,\text{GeV}) = 99 \pm 11\,\text{MeV}$$

FLAG 2010 (preliminary)

Results for quark mass ratios



Phenomenology



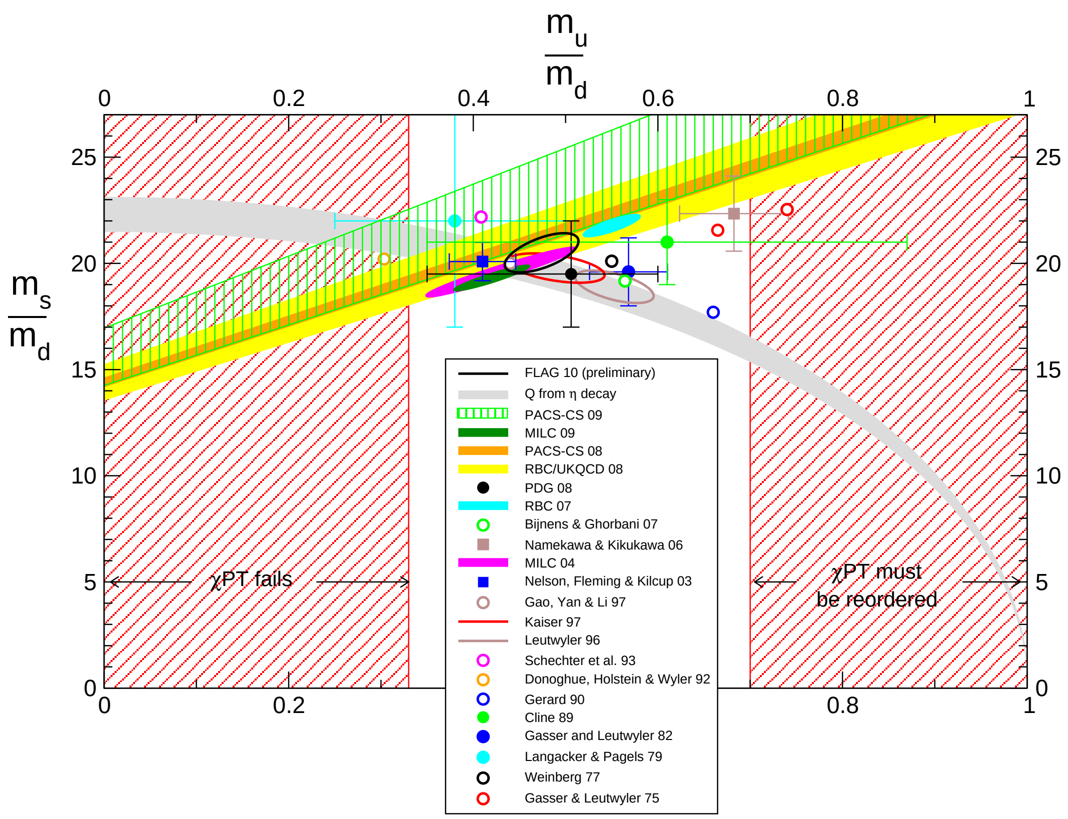
Lattice

$$\frac{m_s}{m_{ud}} = 27.8 \pm 1.0 \qquad \frac{m_u}{m_d} = 0.474 \pm 0.040$$

FLAG 2010 (preliminary)

None of the lattice results is consistent with the "solution" $m_u = 0$ of the strong CP problem

Comparison



$\frac{m_u}{m_d}$
$\frac{m_s}{m_d}$
χPT fails
χPT must be reordered
FLAG 10 (preliminary)
Q from η decay
PACS-CS 09
MILC 09
PACS-CS 08
RBC/UKQCD 08
PDG 08
RBC 07
Bijnens & Ghorbani 07
Namekawa & Kikukawa 06
MILC 04
Nelson, Fleming & Kilcup 03
Gao, Yan & Li 97
Kaiser 97
Leutwyler 96
Schechter et al. 93
Donoghue, Holstein & Wyler 92
Gerard 90
Cline 89
Gasser and Leutwyler 82
Langacker & Pagels 79
Weinberg 77
Gasser & Leutwyler 75

22. V_{us} and V_{ud}

- Experimental sources for V_{us} and V_{ud}:

superallowed nuclear β transitions	$\lvert V_{ud}\rvert$
$K \to \pi\ell\nu$	$\lvert f_+(0)V_{us}\rvert$
$\pi \to \ell\nu$, $\tau \to \pi\nu$	$\lvert V_{ud}\, F_\pi\rvert$
$K \to \ell\nu$, $\tau \to K\nu$	$\lvert V_{us}\, F_K\rvert$
inclusive τ decays	$\lvert V_{us}\rvert$

- Vector current relevant for nuclear β decay is conserved modulo $m_u - m_d$

$\Rightarrow$ analog of $f_+(0)$ is very close to unity

$|V_{ud}| = 0.97425 \pm 0.00022$ Hardy + Towner 2009

- Can determine V_{us} from $K \to \pi\ell\nu$ only if $f_+(0)$ is known. Early determinations were based on χPT prediction for that
- Lattice calculations now provide reliable and precise determination of $f_+(0) \Rightarrow |V_{us}|$
- Results for F_π, F_K do not yet reach sufficient precision, but those for the ratio F_K/F_π do

$\Rightarrow$ $\dfrac{V_{us}}{V_{ud}}$ can be determined from $\dfrac{\Gamma(K \to \ell\nu)}{\Gamma(\pi \to \ell\nu)}$

$\Rightarrow$ can test the Standard Model:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \overset{?}{=} 1$$

$|V_{ub}|$ known well enough, contribution is tiny

- Testing the Standard Model with the lattice data alone

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.002 \pm 0.016$$

- Lattice results for V_{ud} are consistent with the value obtained from nuclear β-decay

$\Rightarrow$ Test sharpens if the two are combined:

$$|V_u|^2 = 1.0000 \pm 0.0007 \qquad f_+(0) \; + V_{ud}$$
$$|V_u|^2 = 0.9999 \pm 0.0007 \qquad F_K/F_\pi + V_{ud}$$

$\Uparrow$ Lattice $\Uparrow$ β-decay

$\Rightarrow$ Can impose $|V_u|^2 = 1$ as a constraint (SM)

	$\|V_{us}\|$	$\|V_{ud}\|$	$f_+(0)$	f_K/f_π
Lattice	0.225(2)	0.9743(4)	0.960(8)	1.193(11)
β decay	0.225(1)	0.9743(2)	0.960(5)	1.192(6)

FLAG review 2010 (preliminary)

- Direct determination of $|V_{us}|$ from τ decay:

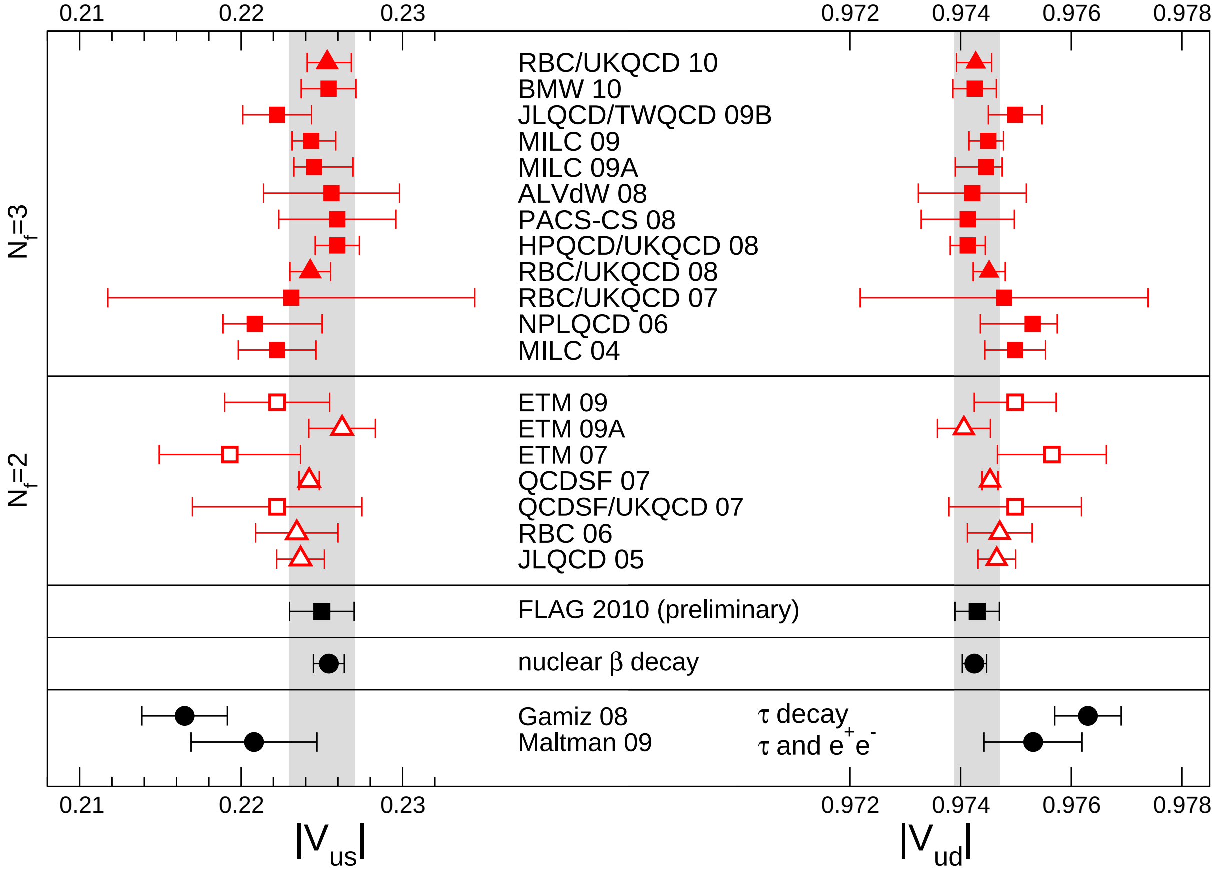
  Sort out the final states in the inclusive decay $\tau \to \nu +$ hadrons:

  $\Gamma = \Gamma(\tau \to \nu + \text{strange hadrons}) + \text{rest}$

  First term dominated by $|V_{us}|^2$, rest by $|V_{ud}|^2$

  Gamiz, Jamin, Pich, Prades, Schwab
  Maltman, Wolfe, Banerjee, Nugent, Roney

Data on $|V_{us}|$ and $|V_{ud}|$ analyzed within the SM:



23. Concluding remarks

- These lectures focused on the low energy properties of the sector with zero baryon number: $N_B = \frac{1}{3}(N_u + N_d + N_s + N_c + N_b + N_t) = 0$. Moreover, only states with $N_c = N_b = N_t = 0$ were discussed.

- There is considerable progress in extending χPT to the sector with $N_B = 1$, as well as to nuclei, where $N_B = 2, 3 \ldots$

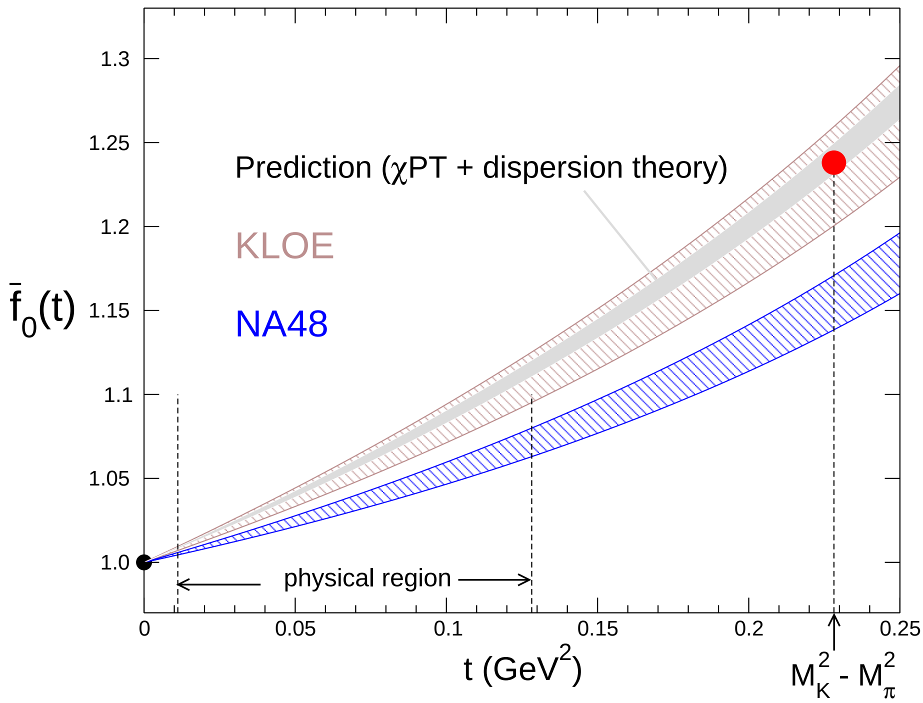
  Hint: ask Prof. Scherer for a course on these developments

- Effective theory for heavy quark bound states

- Mesons with a heavy and a light quark

- Extension from QCD to QCD + QED

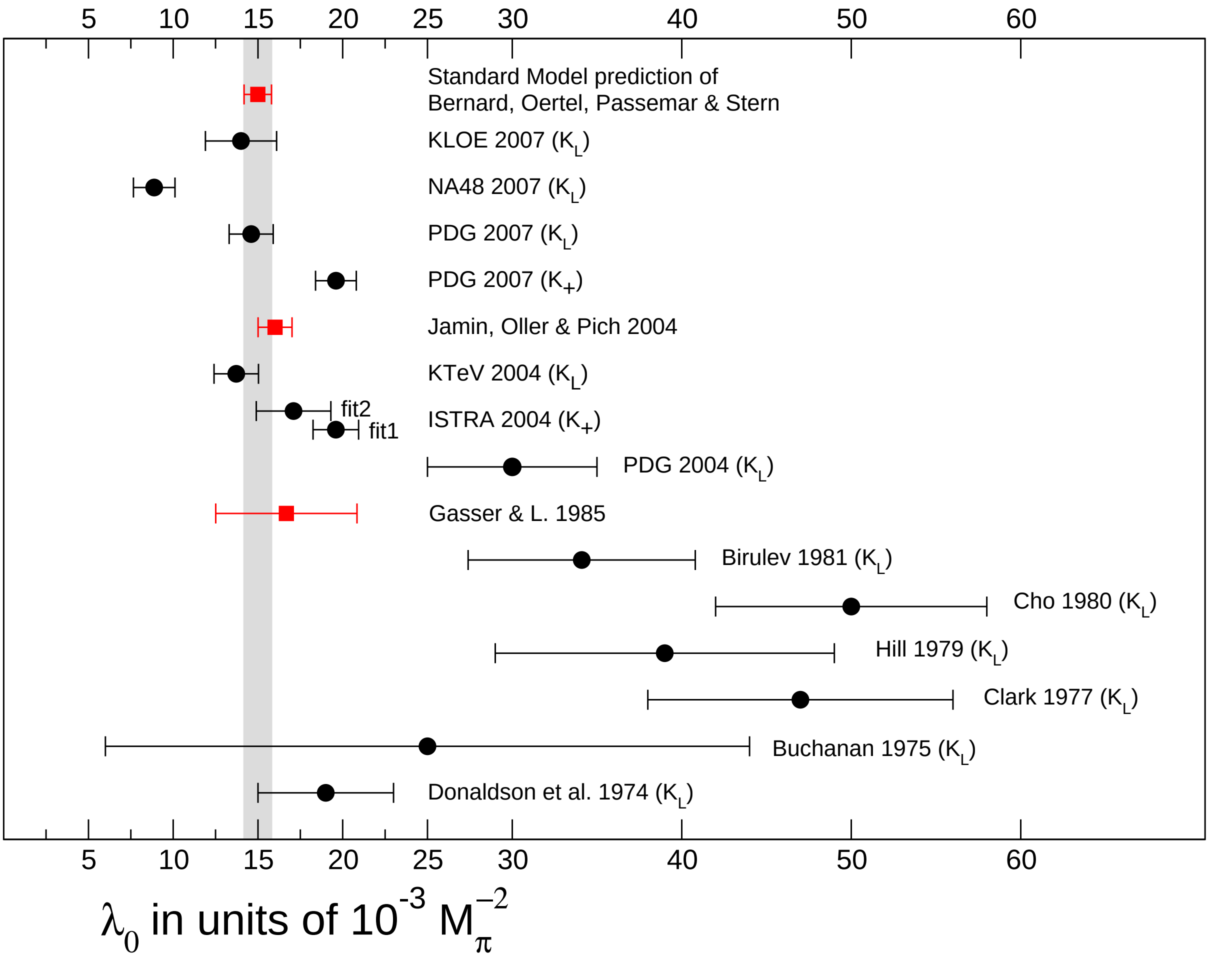
• Puzzle in $K \to \pi\mu\nu$



Plot shows normalized scalar form factor $\bar{f}_0(t) = \dfrac{f_0(t)}{f_0(0)}$

- History of the issue: data on the slope of the scalar form factor

$$f_0(t) = f_0(0)\left\{1 + \lambda_0\, t + \lambda_0'\, t^2 + O(t^3)\right\}$$



- Extend χPT with dispersion theory

  Example: form factors relevant for $K \to \pi\ell\nu$

  $f_0(t) = f_0(0)\left\{1 + \lambda_0\, t + \lambda_0'\, t^2 + \ldots\right\}$

  χPT: $\lambda_0 \leftrightarrow$ NLO, $\lambda_0' \leftrightarrow$ NNLO

  Dispersion theory implies very strong correlation between λ_0 and λ_0'

  Abbas, Ananthanarayan, Caprini, Imsong 2010

- Dispersive analysis of $\pi\pi$ and πK scattering, $\eta \to 3\pi$, . . .

If time permits, I can explain how dispersion theory can be used to extend the χPT result for the $\pi\pi$ scattering lengths to a model-independent prediction for mass and width of the σ meson

Exercises

1. Evaluate the positive frequency part of the massless propagator

$$\Delta^{+}(z,0) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k^0}\, e^{-ikz} \quad , \quad k^0 = |\vec{k}|$$

for $\mathrm{Im}\, z^0 < 0$. Show that the result can be represented as

$$\Delta^{+}(z,0) = \frac{1}{4\pi i z^2}$$

2. Evaluate the d-dimensional propagator

$$\Delta(z,M) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-ikz}}{M^2 - k^2 - i\epsilon}$$

at the origin and verify the representation

$$\Delta(0,M) = \frac{i}{4\pi} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{M^2}{4\pi}\right)^{\frac{d}{2}-1}$$

How does this expression behave when $d \to 4$?

3. Leading order effective Lagrangian:

$$
\begin{aligned}
\mathcal{L}^{(2)} &= \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle + h_0 D_\mu\theta D^\mu\theta \\
D_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu) \\
\chi &= 2\,B\,(s + ip) \\
D_\mu\theta &= \partial_\mu\theta + 2\langle a_\mu\rangle \\
\langle X\rangle &= \mathrm{tr}X
\end{aligned}
$$

- Take the space-time independent part of the external field $s(x)$ to be isospin symmetric (i. e. set $m_u = m_d = m$):

$$s(x) = m\,\mathbf{1} + \tilde{s}(x)$$

- Expand $U = \exp i\,\phi/F$ in powers of $\phi = \vec{\phi}\cdot\vec{\tau}$ and check that, in this normalization of the field ϕ, the kinetic part takes the standard form

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \tfrac{1}{2}M^2\vec{\phi}^2 + \ldots$$

  with $M^2 = 2mB$.

- Draw the graphs for all of the interaction vertices containing up to four of the fields $\phi, v_\mu, a_\mu, \tilde{s}, p, \theta$.

4. Show that the classical field theory belonging to the QCD Lagrangian in the presence of external fields is invariant under

$$\begin{aligned} v'_\mu + a'_\mu &= V_{\mathrm{R}}(v_\mu + a_\mu)V_{\mathrm{R}}^\dagger - i\partial_\mu V_{\mathrm{R}} V_{\mathrm{R}}^\dagger \\ v'_\mu - a'_\mu &= V_{\mathrm{L}}(v_\mu - a_\mu)V_{\mathrm{L}}^\dagger - i\partial_\mu V_{\mathrm{L}} V_{\mathrm{L}}^\dagger \\ s' + i\,p' &= V_{\mathrm{R}}(s + i\,p)V_{\mathrm{L}}^\dagger \\ q'_{\mathrm{R}} &= V_{\mathrm{R}}\, q_{\mathrm{R}}(x) \\ q'_{\mathrm{L}} &= V_{\mathrm{L}}\, q_{\mathrm{L}} \end{aligned}$$

where $V_{\mathrm{R}}, V_{\mathrm{L}}$ are space-time dependent elements of U(3).

5. Evaluate the pion mass to NLO of χPT . Draw the relevant graphs and verify the representation

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\frac{1}{i}\Delta(0, M^2) + O(M^6)$$

6. Start from the symmetry property of the effective action,

$$S_{\mathrm{QCD}}\{v', a', s', p', \theta'\} = S_{\mathrm{QCD}}\{v, a, s, p, \theta\} - \int dx \langle \beta\Omega\rangle,$$

and show that this relation in particular implies the Ward identity

$$\begin{aligned} \partial^x_\mu \langle 0|\, T A^\mu_a(x)\, P_b(y)\,|0\rangle &= -\tfrac{1}{4}\, i\,\delta(x-y)\langle 0|\,\overline{q}\{\lambda_a, \lambda_b\}q\,|0\rangle \\ &\quad + \langle 0|\, T\overline{q}(x)\, i\gamma_5\{m, \tfrac{1}{2}\lambda_a\}q(x)\, P_b(y)\,|0\rangle \end{aligned}$$

$a = 1, \ldots, 8,\ b = 0, \ldots, 8$

7. What is the Ward identity obeyed by the singlet axial current,

$$\partial^x_\mu \langle 0|\, T A^\mu_0(x)\, P_b(y)\,|0\rangle = \ ?$$