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On the main quantum number dependence of the pionium production

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We will use here the notation and equations from Ref. [1]. Neglecting the production of the $\pi^+\pi^-$ atoms with the orbital angular momentum $l > 0$ (suppressed by powers of the $\pi^+\pi^-$ Bohr radius $|a| = 387.5$ fm), the probability to produce a pionium as a result of the two-pion final state interaction (FSI) depends on the main quantum number n as (see Eq. (28) in Ref. [1])

$$w_n \propto (1+\delta_n)|\psi_{n0}^{\rm coul}(0)|^2 \propto (1+\delta_n)/n^3, \tag{1}$$

where $\psi_{n0}^{\rm coul}(0)$ is the pure Coulomb wave function of the $\pi^+\pi^-$ atom at zero separation and the correction factor $(1+\delta_n)$ takes into account the effects of finite-size of the pion production region and the two-pion strong FSI. It can be shown that for two pions produced at a distance in their center-of-mass system much smaller than the Bohr radius $|a|$, the n-dependence of the correction factor is dominated by the renormalization effect of the strong FSI on the two-pion atomic wave function (see Eq. (126) in Ref. [1]):

$$(1+\delta_n) = \left[1 + \phi(n)\frac{2\Re A^{\alpha\alpha}}{n|a|}\right](1+\delta_n'). \tag{2}$$

Here $\phi(n) \approx 3$ and $\Re A^{\alpha\alpha} \approx 0.2$ fm are respectively defined in Eqs. (80), (86) and (115) of Ref. [1]. Thus

$$\phi(n) = 2 + 2n[\ln n - \psi(n)], \tag{3}$$

where the digamma function $\psi(n)$ for the integer argument is given by the recurrence relation:

$$\psi(n+1) = \psi(n) + 1/n, \quad \psi(1) = -C \doteq -0.5772156649. \tag{4}$$

With the increasing n, $\phi(n)$ slowly converges to 3, the first 10 values to 5 digit accuracy being equal to 3.15443, 3.08145, 3.05497, 3.04141, 3.03320, 3.02770, 3.02376, 3.02080, 3.01850, 3.01665. Further,

$$\Re A^{\alpha\alpha} \doteq f_0^{\alpha\alpha} - f_0^{\beta\beta}\frac{(k_\beta^* f_0^{\beta\alpha})^2}{1+(k_\beta^* f_0^{\beta\beta})^2} \approx f_0^{\alpha\alpha} \equiv f_0, \tag{5}$$

where the amplitudes $f_0^{\alpha\alpha'}$ are expressed through the two-pion isoscalar and isotensor s-wave scattering lengths a_0^0 and a_0^2 as (see Eq. (108) in Ref. [1])

$$f_0^{\alpha\alpha} = \frac{2}{3}a_0^0 + \frac{1}{3}a_0^2, \quad f_0^{\alpha\beta} = f_0^{\beta\alpha} = -\frac{\sqrt{2}}{3}(a_0^0 - a_0^2), \quad f_0^{\beta\beta} = \frac{1}{3}a_0^0 + \frac{2}{3}a_0^2 \tag{6}$$

and $k^*_\beta = 35.5$ MeV/c is the π^0 momentum in the channel $\beta = \{\pi^0\pi^0\}$ at the threshold of the channel $\alpha = \{\pi^+\pi^-\}$. Using the values of the scattering lengths from Ref. [2], one has $\Re A^{\alpha\alpha} = 0.18635$ fm.

Taking into account that the factor $(1+\delta'_n)$ is practically independent of n except for a tiny fraction of the pairs containing a pion from η' decay (see the most right panel in Fig. 12 of Ref. [1]), one can write the n-dependence of the pionium production probability in a simple analytical form:

$$w_n \propto \left[1+\phi(n)\frac{2\Re A^{\alpha\alpha}}{n|a|}\right]\frac{1}{n^3} \approx \left(1+\frac{0.3\%}{n}\right)\frac{1}{n^3}, \tag{7}$$

where the approximate equality neglects a weak n-dependence of $\phi(n)$.

In Refs. [3, 4], the effect of the strong interaction on the n–dependence of the pionium wave function has been studied numerically, solving the corresponding Schrödinger equations. Thus, in Ref. [3], the ratio $R_n = \psi_{n0}/\psi^{\rm coul}_{n0}$ and the difference $\Delta R_n = R_1 - R_n$ have been calculated for $n = 1-3$ using an exponential form of the short–range potential. According to Eqs. (83), (90) and (92) of Ref. [1], one has, up to corrections $\mathcal{O}(f_0/a)$ and $\mathcal{O}(r^{*2}/a^2)$:

$$R_n \equiv \frac{\psi_{n0}(r^*)}{\psi^{\rm coul}_{n0}(r^*)} \doteq 1+\frac{f_0}{r^*}, \quad \Delta R_n \equiv R_1 - R_n \doteq \frac{f_0}{|a|}\left\{\phi(1)-\frac{1}{n}\phi(n)\right\}\left(1+\frac{f_0}{r^*}\right). \tag{8}$$

From Fig. 1 of Ref. [3], one can deduce a value of ~ 0.15 fm for the scattering length f_0 to achieve an agreement with the prediction of Eq. (8) for the ratio R_n at $d < r^* \ll |a|$. The differences ΔR_n, presented in Fig. 1 of Ref. [3] for $n = 2$ and 3, are however by a factor 1.6 higher than the corresponding predictions of Eq. (8). For example, for $10^3\Delta R_n$ at $r^* = 8$ fm, $n = 2$ and 3, one can read from this figure the values[1] 1.0 and 1.3 while, Eq. (8) respectively predicts 0.6 and 0.8. This discrepancy may indicate that the calculation error, declared in Ref. [3] to be better than 10^{-4}, was underestimated by a factor of 5.

In Ref. [4], a more refined numerical study of the n–dependence has been done accounting for the second channel ($\pi^0\pi^0$) and extended charges. The hadronic $\pi\pi$ potentials have been chosen to reproduce the phase shifts given by two–loop chiral perturbation theory. The quantity $d_n = n^{3/2}\psi_{n0}/\psi_{10} - 1$ has been calculated for $n = 1-4$. Similar to Eq. (8), one has for $d < r^* \ll |a|$

$$d_n \equiv n^{3/2}\frac{\psi_{n0}(r^*)}{\psi_{10}(r^*)} - 1 \doteq -\frac{f_0}{|a|}\left\{\phi(1)-\frac{1}{n}\phi(n)\right\}, \tag{9}$$

up to corrections $\mathcal{O}(f_0 r^*/a^2)$ and $\mathcal{O}(r^{*2}/a^2)$. The results of numerical calculations presented in Fig. 2 of Ref. [4] are in qualitative agreement with Eq. (9), d_n being almost constant (except for the region of very small r^*) and showing the right n–dependence: $d_n \sim -(1-1/n)$. Similar to Ref. [3], the numerical results for $|d_n|$ are however higher, now by a factor of 2.5, than the predictions of Eq. (9) calculated with $f_0 = 0.2$ fm which should correspond within $\sim 10\%$ to the choice of the potentials in Ref. [4]. Since the presence of the second channel leads to a negligible modification of Eq. (9) ($\Re A^{\alpha\alpha} \approx f_0$) and the correction due to the extended charges is also expected to be negligible ($\sim -\frac{1}{6}\langle r^2\rangle_\pi/a^2$), the discrepancy in the size of the correction d_n has to be attributed to the insufficient calculation accuracy or, to the incorrect matching of the scattering length.

[1] One should correct the figure by interchanging the curves. The author is grateful to O. Voskresenskaya for pointing out this misprint.

References

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