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The lifetimes of hydrogen-like $A_{2\pi}$ atom are basically determined by the spontaneous radiation decay rates unless the orbital momentum equals zero. The lifetime of an *ns*-state is determined by the dominating rate of annihilation connected with the process $\pi^+ + \pi^- \to \pi^0 + \pi^0$, which is about four orders greater than the rate of the radiation decay (see below the data of Table 1) and may be presented as a function of the principal quantum number $\tau_{ns}^{ann} = 3 \cdot 10^{-15} \cdot n^3 / \mathrm{s}$. So, the possibilities exist to observe sufficiently long distances of flight of relativistic $A_{2\pi}$ atoms, only if they appear in states with nonzero angular momentum. We present below the numerical data for the lifetimes of excited states of $A_{2\pi}$ atom, determined by the spontaneous decay rates in comparison with the annihilation rate for *ns*-states.

The general relation for spontaneous radiation transition from an initial excited bound state $|nlm\rangle$ to a final state $|n'l'm'\rangle$ of an isolated $A_{2\pi}$ atom, after integration over emitted-photon wave-vector directions, may be presented, as [1]

$$P_{nlm\to n'l'm'}^{sp} = \frac{4\omega_{nn'}^3}{3c^3}\left|\langle n'l'm'|D_\mu|nlm\rangle\right|^2, \qquad (1)$$

where $c = 137.036$ a. u. is the speed of light, $\omega_{nn'} = E_n - E_{n'}$ is the frequency of emitted photon, $D_\mu = r\,\mathbf{C}_{1\mu}(\mathbf{r}/r)$ is the dipole operator of atom ($\mathbf{r} = r\mathbf{e}_r$ is the position vector of the π^+ meson relative the π^- meson, $\mathbf{C}_{1\mu}(\mathbf{e}_r)$ is the modified spherical function of the position-vector angular variables [2]). The rate (1) of transition to a final state with fixed m' determines also the probability of emission of spontaneous photon with fixed polarization: along the quantization axis ($\mu = 0$), and in a plane, perpendicular to this axis ($\mu = \pm 1$). Generally speaking, these separate probabilities are strongly dependent on the initial-state magnetic quantum number m, except for states 2p and 3p which can decay only into spherically symmetric 1s- and 2s-states.

Meanwhile, for the total decay rate $P_{nlm}^{dec} = \sum_{n'l'm'} P_{nlm\to n'l'm'}^{sp}$ of the state $|nlm\rangle$ the sum over m' (which automatically involves the summation over μ) of expression in the right-hand side of (1) gives an m-independent result. The summations may be performed analytically after integration of the dipole matrix element over angular variables [2]:

$$P_{nlm}^{dec} = \frac{4}{3c^3}\sum_{n'l'm'\mu}\omega_{nn'}^3\left|\langle n'l'|r|nl\rangle\right|^2\frac{2l+1}{2l'+1}\left(C_{l010}^{l'0}\right)^2\left(C_{lm1\mu}^{l'm'}\right)^2 = \frac{4}{3c^3}\sum_{n'l'}\frac{l_>\omega_{nn'}^3}{2l+1}\left|\langle n'l'|r|nl\rangle\right|^2, \qquad (2)$$

where $l_> = (l + l' + 1)/2$ is the greater of the two momentums l and l'. Thus, after calculating sums of Clebsh-Gordan coefficients $C_{lm1\mu}^{l'm'}$ over the final-state magnetic quantum numbers m', only summation over final-state principal and orbital quantum numbers remains in equation (2) determining the total decay rate of an excited state $|nlm\rangle$ and its lifetime $\tau_{nlm}^{sp} = 1/P_{nlm}^{dec}$, both independent of the magnetic quantum number. Consequently, the right-hand side of equation (2) does not change after averaging over the initial-state magnetic quantum number m,

$$P_{nl}^{dec} = \frac{1}{2l+1}\sum_{m=-l}^{m=l} P_{nlm}^{dec} = P_{nlm}^{dec} \qquad (3)$$

So, the calculation of rates (3) is reduced to determining l-independent transition frequencies $\omega_{nn'}$ and l-dependent radial matrix elements $\langle n'l'|r|nl\rangle$. The latter may be performed with the use of

Gordon equations in terms of two Gauss hypergeometric functions ${}_2F_1(a,b;c;z)$ [3] of three parameters and one variable (see, for example [4]). An alternative equation may be proposed, written in terms of the generalized hypergeometric function of two variables and five parameters $F_2(a;b_1,b_2;c_1,c_2;x_1,x_2)$ [3], as follows

$$\langle n'l'|r|nl\rangle = \frac{1}{4}\left(\frac{2n'}{n+n'}\right)^{l+2}\left(\frac{2n}{n+n'}\right)^{l'+2}\frac{(l+l'+3)!}{(2l+1)!(2l'+1)!}\sqrt{\frac{(n+l)!(n'+l')!}{n_r!n'_r!}}\times$$
$$\times F_2\left(l+l'+4;-n_r,-n'_r;2l+2,2l'+2;\frac{2n'}{n+n'},\frac{2n}{n+n'}\right), \tag{4}$$

where $n_r = n-l-1$ is the radial quantum number. We perform calculations in the pionium system of units which differs from the atomic units by the ratio of the reduced mass of the pionium atom to the mass of electron, $m_\pi / m_e \approx 136.6$. This factor enlarges the unit of energy and reduces the unit of length in comparison with those of commonly used atomic system of units. Therefore, the radiation transition probabilities equal to those of hydrogen times 136.6. Thus computed data, presented in **Table 1** for the dipole transitions from excited states of $A_{2\pi}$ atom, are in satisfactory agreement with corresponding data for hydrogen of the book [4]. As is seen from the table, the principal contributions to spontaneous decay rate of *nl*-state come from transitions into the lowest *n'(l-1)*-states. For hydrogen-like states with highest possible angular momentums $l = n-1$ and $l = n-2$ (so-called circular and near-circular states) only transitions into states with lower momentums $l' = l-1 = n-2$ and $l' = n-3$, correspondingly, are allowed. Therefore, these states have the longest lifetimes among all n^2 degenerate substates of the hydrogenic *n*-shell with different angular momentums *l*. Analytical equations may be written for the rates of downward dipole transitions from these long-living states (in atomic units):

$$P^{sp}_{nl\to n-1\,l-1}(l=n-1) = \frac{(2n-1)}{3c^3n^4(n-1)^2}\left(\frac{4n(n-1)}{(2n-1)^2}\right)^{2n}\frac{m_\pi}{m_e} \tag{5}$$

$$P^{sp}_{nl\to n-1\,l-1}(l=n-2) = \frac{(2n-1)^3(n-2)}{12c^3n^6(n-1)^3}\left(\frac{4n(n-1)}{(2n-1)^2}\right)^{2n}\frac{m_\pi}{m_e} \tag{6}$$

$$P^{sp}_{nl\to n-2\,l-1}(l=n-2) = \frac{8(n-1)^4}{3c^3n^6(n-2)^4}\left(\frac{n(n-2)}{(n-1)^2}\right)^{2n}\frac{m_\pi}{m_e} \tag{7}$$

The factor $1.6065\cdot10^{10}$ s^{-1}/a.u. transforms the numerical values given by these expressions in atomic units into the rates in the units of s^{-1}. With inclusion of the mass ratio into this factor, $m_\pi/m_e\cdot1.6065\cdot10^{10}$ s^{-1}/a.u.=$2.1945\cdot10^{12}$ s^{-1}/a.u., equation (5) gives an asymptotic (for $n\to\infty$) expression for the lifetimes of the circular orbits of the $A_{2\pi}$ atom (in picoseconds)

$$\tau^{sp}_{nl}(l=n-1) \approx 0.68354\cdot n^4(n-1)\ \text{ps}. \tag{8}$$

Numerical values given by asymptotic approximation (8) differ from those of the exact result (5) by about 0.1% for n=10, by 0.001% for *n*=100.

References

[1] Sobelman I.I. 1996, "Atomic spectra and radiative transitions" (Berlin: Springer). (Vvedenie v teoriyu atomnykh spektrov", 1977, Moscow: Nauka).
[2] Varshalovich D.A., Moskalev A.N. and Khersonskii V.K. 1988 "Quantum theory of angular momentum" (Singapore: World Scientific)
[3] Bateman H. and Erdelyi, 1953, "Higher transcendental functions" (NY: McGraw-Hill)
[4] Bethe H.A. and Salpeter, 1957, "Quantum mechanics of one- and two-electron atoms" (Berlin: Springer-Verlag).

Table 1. Rates of dipole radiation decay, spontaneous radiation lifetimes τ_{nl}^{sp} of excited states of $A_{2\pi}$ atom and ns-state-annihilation lifetimes τ_{ns}^{ann} connected with the process "$\pi^+ + \pi^- \to \pi^0 + \pi^0$".

nl	$n'l'$	$P_{nl\to n'l'}^{sp}/(10^9/\mathrm{s})$	$P_{nl}^{dec}/(10^9/\mathrm{s})$	$\tau_{nl}^{sp} = 1/P_{nl}^{dec}$/ns	$\tau_{ns}^{ann} = 3\cdot 10^{-6} n^3$ / ns
2p	1s	85.62	85.62	0.01168	
3s	2p	0.8629	0.8629	1.1589	$8.1\cdot 10^{-5}$
3p	1s	22.86			
	2s	3.068	25.93	0.03857	
3d	2p	8.836	8.836	0.11317	
4s	2p	0.35236			
	3p	0.25085	0.60322	1.6578	$1.92\cdot 10^{-4}$
4p	1s	9.319			
	2s	1.321			
	3s	0.419	11.060		
	3d	0.0475	11.107	0.09003	
4d	2p	2.819			
	3p	0.962	3.781	0.2645	
4f	3d	1.884	1.884	0.5307	
5s	2p	0.17612			
	3p	0.12365			
	4p	0.08816	0.38793	2.5778	$3.75\cdot 10^{-4}$
5p	1s	4.6982			
	2s	0.67630			
	3s	0.22383			
	4s	0.10075	5.6991		
	3d	0.02044			
	4d	0.02576	5.7453	0.17406	
5d	2p	1.2882			
	3p	0.46352			
	4p	0.20307			
	4f	0.00690	1.9617	0.50977	
5f	3d	0.62079			
	4d	0.35322	0.97401	1.02668	
5g	4f	0.58143	0.58143	1.71989	
6s	2p	0.10045			
	3p	0.06932			
	4p	0.04896			
	5p	0.03665	0.25538	3.91567	$4.48\cdot 10^{-4}$
6p	1s	2.69634			
	2s	0.39066			
	3s	0.13053			
	4s	0.06090			
	5s	0.03321	3.31164		
	3d	0.01069			
	4d	0.01287			
	5d	0.01311	3.34831	0.29866	
6d	2p	0.70318			
	3p	0.25664			
	4p	0.11784			
	5p	0.06143			
	4f	0.00293			
	5f	0.00534	1.14737	0.87156	

Table 1 (continuation).

nl	$n'l'$	$P^{sp}_{nl\to n'l'}$/(10^9/s)	P^{dec}_{nl}/(10^9/s)	$\tau^{sp}_{nl}=1/P^{dec}_{nl}$/ns	$\tau^{ann}_{ns}=3\cdot10^{-6}n^3$/ns
6f	3d	0.29331			
	4d	0.17590			
	5d	0.09885			
	5g	0.00155	0.56960	1.75560	
6g	4f	0.18762			
	5f	0.15112	0.33874	2.95211	
6h	5g	0.22480	0.22480	4.44831	
7s	2p	0.06269			
	3p	0.04275			
	4p	0.02971			
	5p	0.02210			
	6p	0.01729	0.17454	5.72924	$1.035\cdot10^{-3}$
7p	1s	1.68954			
	2s	0.24562			
	3s	0.08235			
	4s	0.03865			
	5s	0.02174			
	6s	0.01338	2.09129		
	3d	0.00634			
	4d	0.00737			
	5d	0.00725			
	6d	0.00698	2.11922	0.47187	
7d	2p	0.42793			
	3p	0.15712			
	4p	0.07295			
	5p	0.03967			
	6p	0.02329	0.72096		
	4f	0.00154			
	5f	0.00261			
	6f	0.00345	0.72856	1.37257	
7f	3d	0.16497			
	4d	0.10032			
	5d	0.05920			
	6d	0.03545	0,35994		
	5g	0.00064			
	6g	0.00152	0.36209	2.76174	
7g	4f	0.08827			
	5f	0.07492			
	6f	0.05144	0.21463		
	6h	0.00046	0.21509	4.64932	
7h	5g	0.06953			
	6g	0.07275	0.14227	7.02867	
7i	6h	0.10124	0.10124	9.87709	

Table 1 (continuation).

nl	$n'l'$	$P^{sp}_{nl\to n'l'}/(10^9/\mathrm{s})$	$P^{dec}_{nl}/(10^9/\mathrm{s})$	$\tau^{sp}_{nl}=1/P^{dec}_{nl}$/ns	$\tau^{ann}_{ns}=3\cdot10^{-6}n^3$/ns
8s	2p	0.04174			
	3p	0.02825			
	4p	0.01940			
	5p	0.01421			
	6p	0.01108			
	7p	0.00898	0.12366	8.08678	$1.54\cdot10^{-3}$
8p	1s	1.12822			
	2s	0.16435			
	3s	0.05519			
	4s	0.02593			
	5s	0.01464			
	6s	0.00927			
	7s	0.00620	1.40379		
	3d	0.00408			
	4d	0.00464			
	5d	0.00444			
	6d	0.00416			
	7d	0.00393	1.42503	0.70174	
8d	2p	0.28050			
	3p	0.10330			
	4p	0.04817			
	5p	0.02642			
	6p	0.01612			
	7p	0.01031	0.48482		
	4f	0.00092			
	5f	0.00149			
	6f	0.00186			
	7f	0.00218	0.49127	2.03556	
8f	3d	0.10304			
	4d	0.06301			
	5d	0.03763			
	6d	0.02359			
	7d	0.01505	0.24232		
	5g	0.00033			
	6g	0.00072			
	7g	0.00116	0.24452	4.08967	
8g	4f	0.04979			
	5f	0.04284			
	6f	0.03092			
	7f	0.02099	0.14454		
	6h	0.00018			
	7h	0.00053	0.14525	6.88471	
8h	5g	0.03196			
	6g	0.03523			
	7g	0.02859	0.09578		
	7i	0.00016	0.09594	10.4228	
8i	6h	0.02973			
	7h	0.03835	0.06809	14.6872	
8j	7i	0.05091	0.05091	19.6431	