Chapter 1

Electrostatics

The computation of the electrostatic field and potential is a two-step process:

1. the calculation of charges (per unit length) on the wires and of a reference potential reproducing the wire-potentials at the wire-surfaces in step 2. Various boundary conditions may have to be satisfied. The equations solved in this step are known as capacitance equations.

2. the summation of the contributions of each wire to the field and potential at any given position, using the charges calculated step 1.

Both steps require an expression for the electrostatic potential, which will be derived for the various situations in the first part of this chapter. It will also be shown in this chapter that most cylindrical geometries can be reduced to Cartesian geometries by applying suitable coordinate transformations.

1.1 Notation

We will use the following notation in this chapter:

- n: number of wires,
- $z_j = (x_j, y_j)$: position of wire j,
- r_j: radius of wire j,
- q_j: charge of wire j,
- V_j: surface potential of wire j,
- X_1, X_2: positions of the planes at constant x,
- Y_1, Y_2: positions of the planes at constant y,
- $W(z) = W(x, y)$: complex potential at z or at (x, y),

Planes	Not periodic	x-periodic	y-periodic	doubly periodic
none	A	B1x	B1y	C1
1 x	A mirror	B2x	B1y mirror	C2y
2 x	B2x	B2x	C2y	C2y
1 y	A mirror	B1x mirror	B2y	C2x
1 x, 1 y	A mirror	B2x mirror	B2y mirror	C3
2 x, 1 y	B2x mirror	B2x mirror	C3	C3
2 y	B2y	C2x	B2y	C2x
1 x, 2 y	B2y mirror	C3	B2y mirror	C3
2 x, 2 y	C3	C3	C3	C3

Table 1.1: Table of the cell types

- $\phi(z) = V(z) = ReW(z)$: potential at z induced by a unit charge at the origin,
- s_{x}, s_{y}: x and y periodicities.

1.2 Cell types

Garfield should be able to handle all rectangular and some cylindrical 2-dimensional cells not involving more than 2 equipotential planes in either the x (r)- or the y (ϕ)-direction. Repetition of the cell in the x (r)- and/or the y-direction is allowed. Table 1.2 shows the names used in the program for each of the potentials; the mention 'mirror' means that mirror charges are introduced in the field calculations.

The positions of the charges and their mirror images for each of the potentials is shown in Fig 1.2. As can be seen from the plot, the A potentials are for single charges, the B potentials for rows and the C potentials for grids of charges.

No distinction will be made between the physical charges and the (mathematical) mirror charges in the rest of this chapter unless otherwise stated.

1.3 Isolated charges (type A)

The potential for a line charge at the origin is:

$$\phi(x,y) = \phi_0 + \frac{1}{2\pi\epsilon_0}\log(\sqrt{x^2+y^2}/r_0)$$

where ϕ_0 is the potential at a distance r_0 from the origin. Usually, r_0 is chosen to be the radius of the wire in which case ϕ_0 equals the surface potential.

1.4 Rows of charges (types B1x, B1y, B2x and B2y)

We will use the following scheme to find these potentials:

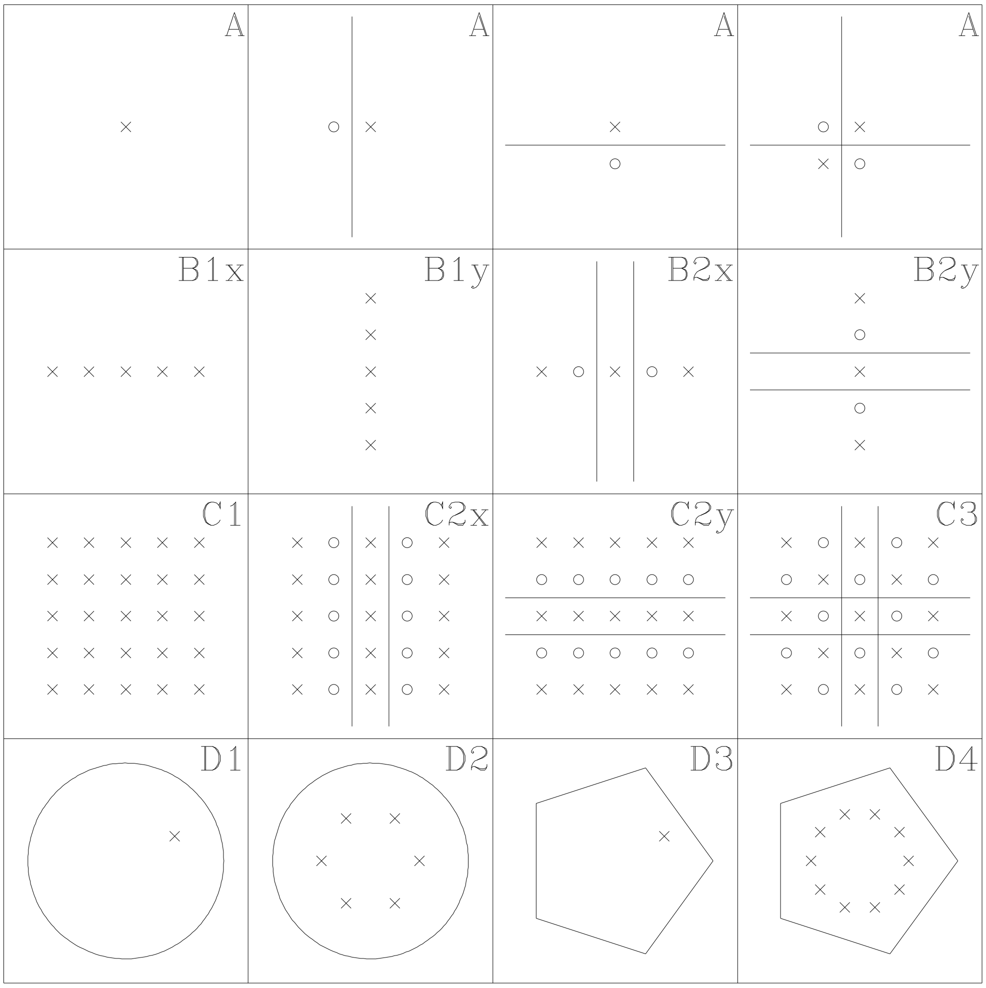

Figure 1.1: Schematic arrangement of the charges

1. Determine a complex entire function $F^+(z)$ satisfying the following requirements:
   (a) $F^+(z)$ has simple zeros at the positions of the wires and their even images,
   (b) the coefficient of z in the expansion of $F^+(z)$ around $z = 0$ equals 1, as required by the Maxwell equations.
2. Determine a similar function $F^-(z)$ for the odd images. Set $F^-(z) = 1$ if these do not exist.
3. Define the function $F(z)$:

$$F(z) = F^+(z)/F^-(z)$$

4. Then, a potential solving our problem is:

$$\phi(x, y) = \frac{-1}{2\pi\epsilon_0}\mathrm{Re}\,\log F(z)$$

   and after some algebra the corresponding electric field turns out to be:

$$E_x - iE_y = \frac{1}{2\pi\epsilon_0}\frac{F'(z)}{F(z)}$$

We will first of all deal with type B1x. The obvious choice for an F which has a zero at each of the replicas of the wire is the following:

$$F_{\mathrm{B1x}}(z) = (z - z_0)\prod_{n=1}^{\infty}\left(1 - \frac{z - z_0}{ns_{\mathrm{x}}}\right)\left(1 + \frac{z - z_0}{ns_{\mathrm{x}}}\right)$$

Using the sine-product, see for instance [5]:

$$\sin(z) = z\prod_{n=1}^{\infty} 1 - \left(\frac{z}{n\pi}\right)^2$$

the following simple expression follows:

$$F_{\mathrm{B1x}}(z) = \sin\left(\pi\frac{z - z_0}{s_{\mathrm{x}}}\right)$$

The hyperbolic analog of the sine-product:

$$\sinh(z) = z\prod_{n=1}^{\infty} 1 + \left(\frac{z}{n\pi}\right)^2$$

leads to the B1y potential:

$$F_{\mathrm{B1y}}(z) = \sinh\left(\pi\frac{z - z_0}{s_{\mathrm{x}}}\right)$$

The B2x and B2y potentials are mere superpositions of the B1x and B1y potentials:

$$F_{\mathrm{B2x}} = \frac{\sin\left(\pi \frac{z-z_0}{2(X_2-X_1)}\right)}{\sin\left(\pi \frac{z-z_0'}{2(X_2-X_1)}\right)}$$

where $z_0' = 2(X_2 - X_1) + iy$

$$F_{\mathrm{B2y}} = \frac{\sinh\left(\pi \frac{z-z_0}{2(X_2-X_1)}\right)}{\sinh\left(\pi \frac{z-z_0'}{2(X_2-X_1)}\right)}$$

where $z_0' = x + 2i(Y_2 - Y_1)$

1.5 Electrostatic field of a doubly periodic wire array

(Contributed by G. A. Erskine)

1.5.1 Specification of the array

We consider a doubly periodic array of thin wires, the array consisting of replicas of a basic rectangular cell defined by $0 \le x \le s_{\mathrm{x}}$, $0 \le y \le s_{\mathrm{y}}$. This cell contains n wires, where wire j is characterised by:

Position	...	$z_j = x_j + iy_j,$
Radius	...	$r_j,$
Potential	...	$V_j.$

Wires identical to wire j ($j = 1, 2, \cdots, n$) are situated at:

$$z_j + \lambda s_{\mathrm{x}} + i\mu s_{\mathrm{y}} \qquad (\lambda, \mu = 0, \pm 1, \pm 2, \cdots).$$

1.5.2 The thin-wire potential approximation

We shall obtain a potential function $V(z)$ which satisfies the following conditions:

$$\frac{\partial^2 V(z)}{\partial x^2} + \frac{\partial^2 V(z)}{\partial y^2} = 0, \qquad z \neq z_j,\ j = \pm 1, \pm 2, \cdots, \pm n. \tag{1.1}$$

$$V(z_k + r_k e^{i\phi}) = V_k + \imath(r_k), \qquad 0 \le \phi \le 2\pi,\ k = 1, 2, \cdots, n. \tag{1.2}$$

$$V(z + s_{\mathrm{x}}) = V(z), \qquad \text{for all } z. \tag{1.3}$$

$$V(z + is_{\mathrm{y}}) = V(z), \qquad \text{for all } z. \tag{1.4}$$

The exact potential function would be defined by the same conditions without any term $\wr(r_k)$ on the right-hand side of (1.2). To simplify the formulae, electrostatic units are used.

We define $V(z) = \operatorname{Re} W(z)$ where

$$W(z) = \sum_{j=1}^{n} q_j \left\{ -2 \log \vartheta_1 \left[\frac{\pi}{s_\mathrm{x}}(z - z_j), e^{-\pi s_\mathrm{y}/s_\mathrm{x}} \right] + i \frac{4\pi}{s_\mathrm{x} s_\mathrm{y}} y_j z \right\} + c \tag{1.5}$$

with the theta function ϑ_1 [8] defined by

$$\vartheta_1(\zeta, p) = 2p^{1/4} \sum_{m=0}^{\infty} -1^m p^{m(m+1)} \sin(2m+1)\zeta, \tag{1.6}$$

and with the $n{+}1$ real constants q_j $(j = 1, 2, \cdots, n)$ and c determined by the system of equations

$$\sum_{j=1}^{n} \left\{ -2 \log \left| \vartheta_1 \left[\frac{\pi}{s_\mathrm{x}}(z_i - z_j), e^{-\pi s_\mathrm{y}/s_\mathrm{x}} \right] \right| - \frac{4\pi y_i y_j}{s_\mathrm{x} s_\mathrm{y}} \right\} q_j + c = V_i, \quad (i = 1, 2, \cdots, n), \tag{1.7}$$

$$\sum_{j=1}^{n} q_j = 0. \tag{1.8}$$

In (1.7) we use the convention that, for the terms with $i = j$,

$$\vartheta_1 \left[\frac{\pi}{s_\mathrm{x}}(z_i - z_j), p \right] = \frac{\pi r_i}{s_\mathrm{x}} \vartheta_1'(0, p) \tag{1.9}$$

$$= \frac{\pi r_i}{s_\mathrm{x}} 2p^{1/4} \sum_{m=0}^{\infty} (-1)^m (2m+1) p^{m(m+1)} \tag{1.10}$$

The coefficient q_j in (1.5) is the charge per unit length on wire j.

Condition (1.1) is satisfied because $V(z)$ is the real part of a function (1.5) which is analytic everywhere except at the points z_j $(j = 1, 2, \cdots, n)$. Condition (1.2) follows from (1.7) and (1.9). Condition (1.3) is an immediate consequence of (1.5) and (1.6).

To show that condition (1.4) is satisfied, define

$$w_j(z) = -2 \log \vartheta_1 \left[\frac{\pi}{s_\mathrm{x}}(z - z_j), e^{-\pi s_\mathrm{y}/s_\mathrm{x}} \right] + i \frac{4\pi}{s_\mathrm{x} s_\mathrm{y}} y_j z.$$

Then, from the quasi-periodicity of the function ϑ_1 [9], neglecting integral multiples of $2\pi i$,

$$w_j(z + is_\mathrm{y}) = w_j(z) + i4\pi \frac{z - x_j}{s_\mathrm{x}} - 2\pi \frac{s_\mathrm{y}}{s_\mathrm{x}},$$

and hence

$$\operatorname{Re} w_j(z + is_\mathrm{y}) = \operatorname{Re} w_j(z) - \frac{2\pi}{s_\mathrm{x}}(2y + s_\mathrm{y})$$

Therefore, using (1.8),

$$\operatorname{Re} W(z + is_{\mathrm{y}}) = \operatorname{Re} \sum_{j=1}^{n} q_j w_j(z + is_{\mathrm{y}}) + c \tag{1.11}$$

$$= \operatorname{Re} \sum_{j=1}^{n} q_j w_j(z) + c \tag{1.12}$$

$$= \operatorname{Re} W(z). \tag{1.13}$$

1.5.3 Alternative expression for the thin-wire potential

By considering the array obtained by rotating the original array through $90°$;, we see that the real part of the function obtained from $W(z)$ by replacing z by iz, z_j by iz_j $(j = 1, 2, \cdots, n)$, and interchanging s_{x} and s_{y}, also satisfies the conditions (1.1) through (1.4). We thus obtain the alternative expression:

$$W(z) = \sum_{j=1}^{n} q_j \left\{ -2 \log \vartheta_1 \left[\frac{i\pi}{s_{\mathrm{y}}}(z - z_j), e^{-\pi s_{\mathrm{x}}/s_{\mathrm{y}}} \right] - \frac{4\pi}{s_{\mathrm{x}} s_{\mathrm{y}}} x_j z \right\} + c, \tag{1.14}$$

where q_j $(j = 1, 2, \cdots, n)$ and c are determined by

$$\sum_{j=1}^{n} \left\{ -2 \log \left| \vartheta_1 \left[\frac{i\pi}{s_{\mathrm{y}}}(z_i - z_j), e^{-\pi s_{\mathrm{x}}/s_{\mathrm{y}}} \right] \right| - \frac{4\pi x_i x_j}{s_{\mathrm{x}} s_{\mathrm{y}}} \right\} q_j + c = V_i, \quad (i = 1, 2, \cdots, n), \tag{1.15}$$

$$\sum_{j=1}^{n} q_j = 0. \tag{1.16}$$

The parameter p in the series (1.6) has the value $e^{-\pi s_{\mathrm{y}}/s_{\mathrm{x}}}$ when the expression (1.5) is used for $W(z)$, and the value $e^{-\pi s_{\mathrm{x}}/s_{\mathrm{y}}}$ when expression (1.14) is used. We therefore adopt the following rule:

- If $s_{\mathrm{x}} \leq s_{\mathrm{y}}$, compute $W(z)$ from (1.5).
- If $s_{\mathrm{x}} > s_{\mathrm{y}}$, compute $W(z)$ from (1.14).

This rule ensures that, for all values of s_{x} and s_{y}, we have $p \leq e^{-\pi} = 0.043...$.

1.5.4 Field intensity

The components (E_x, E_y) of the electrostatic field intensity vector at z are given by:

$$E_x = -\frac{\partial V}{\partial x} = -\operatorname{Re} W'(z), \tag{1.17}$$

$$E_y = -\frac{\partial V}{\partial y} = +\operatorname{Im} W'(z), \tag{1.18}$$

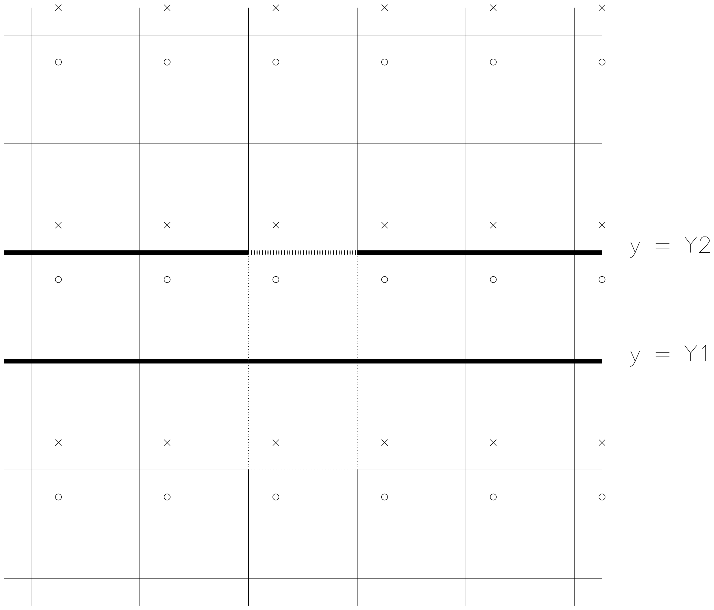

Figure 1.2: Periodic wire array between grounded parallel planes

where, depending upon whether we use (1.5) or (1.14) for $W(z)$,

$$W'(z) = \sum_{j=1}^{n} q_j \left\{ -\frac{2\pi}{s_\mathrm{x}} \frac{\vartheta_1'[\frac{\pi}{s_\mathrm{x}}(z - z_j), e^{-\pi s_\mathrm{y}/s_\mathrm{x}}]}{\vartheta_1[\frac{\pi}{s_\mathrm{x}}(z - z_j), e^{-\pi s_\mathrm{y}/s_\mathrm{x}}]} + i\frac{4\pi}{s_\mathrm{x} s_\mathrm{y}} y_j \right\}$$

or

$$W'(z) = \sum_{j=1}^{n} q_j \left\{ -i\frac{2\pi}{s_\mathrm{y}} \frac{\vartheta_1'[\frac{i\pi}{s_\mathrm{y}}(z - z_j), e^{-\pi s_\mathrm{x}/s_\mathrm{y}}]}{\vartheta_1[\frac{i\pi}{s_\mathrm{y}}(z - z_j), e^{-\pi s_\mathrm{x}/s_\mathrm{y}}]} - \frac{4\pi}{s_\mathrm{x} s_\mathrm{y}} x_j \right\}.$$

1.5.5 Periodic wire array between parallel electrodes

The complex potential of an array of wires which is periodic in the direction of one of the axes, and is bounded by two parallel zero-potential planes, is identical (in the region between the planes) to that of a doubly periodic array whose basic cell contains the original wires together with their reflections (with reversed sign for the wire potential) in one of the planes. If the planes are not at zero potential it is merely necessary to add a term linear in z.

Consider an array consisting of replicas in the x-direction of a group of n wires lying between zero-potential planes situated at $y = Y_1$ and $y = Y_2$. Let the wire positions be z_j and the wire potentials V_j $(j = 1, 2, \cdots, n)$, and let the x-periodicity be s_{x}. If $\overline{z}$ is the complex conjugate of z, the reflection z_j' of z_j in the plane $y = Y_1$ is given by

$$z_j' = \overline{z}_j + i2Y_1.$$

We now define a y-periodicity

$$s_{\mathrm{y}} = 2(Y_2 - Y_1),$$

and consider the doubly periodic array with periods $(s_{\mathrm{x}}, s_{\mathrm{y}})$ consisting of replicas of a basic cell (enclosed by the broken line in Fig 1.5.5) containing $2n$ wires:

Position	z_j	z_j'
Potential	V_j	$-V_j$
Charge	q_j	$-q_j$

From the symmetry of the positive and negative charges with respect to each of the planes $y = Y_1$ and $y = Y_2$ (Fig 1.5.5) it follows that these planes are at zero potential, and hence that the field of the doubly periodic array is the same as that of the original array. Using (1.5) and (1.14) respectively, we obtain the following expressions for the complex potential:

Case 1. $s_{\mathrm{x}} \leq 2(Y_2 - Y_1)$.

$$W(z) = \sum_{j=1}^{n} q_j \left\{ w_{\mathrm{a}}(z - z_j) - w_{\mathrm{a}}(z - z_j') + i\frac{8\pi}{s_{\mathrm{x}} s_{\mathrm{y}}}(y_j - Y_1)z \right\}, \tag{1.19}$$

where

$$w_{\mathrm{a}}(z) = -2\log\vartheta_1\left[\frac{\pi z}{s_{\mathrm{x}}}, e^{-\pi s_{\mathrm{y}}/s_{\mathrm{x}}}\right]. \tag{1.20}$$

Case 2. $s_{\mathrm{x}} > 2(Y_2 - Y_1)$.

$$W(z) = \sum_{j=1}^{n} q_j \left\{ w_{\mathrm{b}}(z - z_j) - w_{\mathrm{b}}(z - z_j') \right\}, \tag{1.21}$$

where

$$w_{\mathrm{b}}(z) = -2\log\vartheta_1\left[\frac{i\pi z}{s_{\mathrm{y}}}, e^{-\pi s_{\mathrm{x}}/s_{\mathrm{y}}}\right]. \tag{1.22}$$

In both cases the q_j are determined by the system of linear equations

$$\sum_{j=1}^{n} \operatorname{Re} W(z_i) q_j = V_i, \qquad (i = 1, 2, \cdots, n), \tag{1.23}$$

where the convention defined by (1.9) is used for the terms with $i = j$.

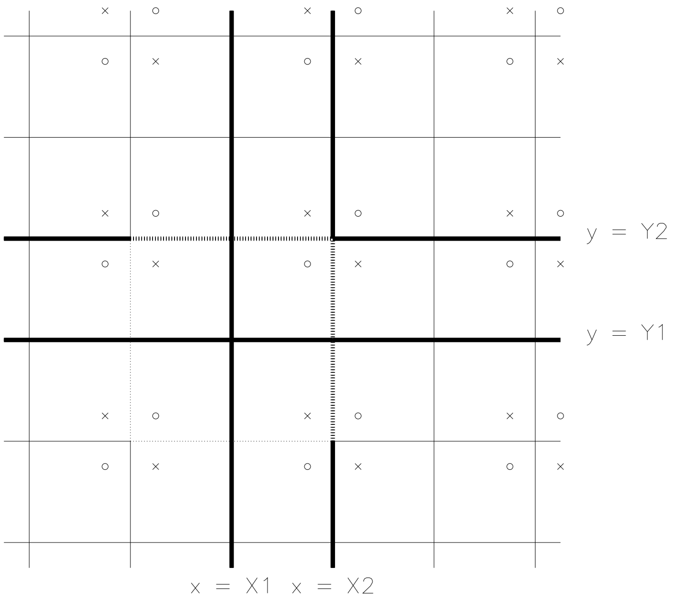

Figure 1.3: Wire array in a grounded rectangular tube

Note that the sum $\sum(q_j + q'_j)$ of the $2n$ charges in the unit cell of the doubly periodic array is equal to zero, but that the sum $\sum q_j$ of the n physical charges is not necessarily equal to zero.

If, instead of being at zero potential, the planes $y = Y_1$ and $y = Y_2$ are at potentials v_1 and v_2 respectively, it is necessary to add to the complex potential W(z) a term representing the superimposed uniform field, namely

$$\frac{(Y_2 v_1 - Y_1 v_2) + i(v_1 - v_2)z}{Y_2 - Y_1}.$$

[Expressions equivalent to (1.19) through (1.22) were obtained by Buchholz [1] by direct summation of the contributions of the individual wires and of their multiple reflections in the planes.]

1.5.6 Wire array inside a rectangular tube

We consider n wires with positions z_j and potentials V_j, $(j = 1, 2, \cdots, n)$ lying inside a rectangular zero-potential tube defined by $X_1 \leq x \leq X_2, Y_1 \leq y \leq Y_2$. The complex potential inside the tube is the same as that of a two-dimensional periodic array of image charges having the symmetry shown in Fig 1.5.6. The periods of this two-dimensional array are

$$s_{\mathrm{x}} = 2(X_2 - X_1), \tag{1.24}$$

$$s_{\mathrm{y}} = 2(Y_2 - Y_1), \tag{1.25}$$

and the basic cell (enclosed by the broken line in Fig 1.5.6) contains $4n$ wires:

Position	Potential	Charge	Description
$z_j^{(0)} = z_j$	V_j	q_j	Physical wire
$z_j^{(1)} = \overline{z_j} + i2Y_1$	$-V_j$	$-q_j$	Reflection in the line $y = Y_1$
$z_j^{(2)} = -z_j + 2(X_1 + iY_1)$	V_j	q_j	Reflection in the point $X_1 + iY_1$
$z_j^{(3)} = -\overline{z_j} + 2X_1$	$-V_j$	$-q_j$	Reflection in the line $x = X_1$

On defining, in terms of (1.20) and (1.22),

$$W(z) = w_{\mathrm{a}}(z) \qquad \text{if } X_2 - X_1 \le Y_2 - Y_1, \tag{1.26}$$

$$W(z) = w_{\mathrm{b}}(z) \qquad \text{if } X_2 - X_1 > Y_2 - Y_1, \tag{1.27}$$

we obtain the required complex potential:

$$W(z) = \sum_{j=1}^{n} q_j \left\{ W(z - z_j^{(0)}) - W(z - z_j^{(1)}) + W(z - z_j^{(2)}) - W(z - z_j^{(3)}) \right\}, \tag{1.28}$$

where the q_j are determined by a system of linear equations (1.23) in which $W(z)$ is computed from (1.28).

1.5.7 Computational considerations

Because the expressions for $W(z)$ either assume the relation $\sum q_j = 0$ or depend only on the ratio of two ϑ_1 functions, the constant $2p^{1/4}$ in (1.6) makes no contribution. The series which need to be evaluated numerically are therefore of the form

$$\sum_{m=0}^{\infty} (-1)^m p^{m(m+1)} \sin(2m+1)\zeta \qquad \text{[for } W \text{ and } W'\text{]}, \tag{1.29}$$

$$\sum_{m=0}^{\infty} (-1)^m (2m+1) p^{m(m+1)} \cos(2m+1)\zeta \qquad \text{[for } W' \text{ only]}, \tag{1.30}$$

where

$$p = e^{-\pi s_{\mathrm{y}}/s_{\mathrm{x}}}, \quad \zeta = \frac{\pi}{s_{\mathrm{x}}}(z - z_j) \quad \text{when } s_{\mathrm{x}} \le s_{\mathrm{y}}, \tag{1.31}$$

$$p = e^{-\pi s_{\mathrm{x}}/s_{\mathrm{y}}}, \quad \zeta = \frac{i\pi}{s_{\mathrm{y}}}(z - z_j) \quad \text{when } s_{\mathrm{x}} > s_{\mathrm{y}}. \tag{1.32}$$

If terms with $m \geq M$ are neglected when evaluating (1.29), the relative error in the sum is bounded approximately by

$$\rho_M = p^{M(M+1)} left| \frac{\sin(2M+1)\zeta}{\sin\zeta} right| \tag{1.33}$$

$$\leq (2M+1)p^{M(M+1)} e^{2M|\mathrm{Im}\,\zeta|}. \tag{1.34}$$

Assuming, as we may without loss of generality, that $|x - x_j| \leq s_{\mathrm{x}}$ and $|y - y_j| \leq s_{\mathrm{y}}$, we find for both (1.5) and (1.14) that $e^{|\mathrm{Im}\,\zeta|} \leq 1/p$. Hence,

$$\rho_M \leq (2M+1)p^{M(M-1)} \tag{1.35}$$

$$\leq (2M+1)e^{-\pi M(M-1)}. \tag{1.36}$$

In the same way, we may show that the relative error resulting from the neglect of terms with $m \geq M$ in the evaluation of (1.30) is bounded approximately by $\sigma_M = (2M+1)\rho_M$. Setting $M = 3$, and using $p \leq e^{-\pi}$,

$$\rho_3 \leq 4.6 \times 10^{-8}, \tag{1.37}$$

$$\sigma_3 \leq 3.2 \times 10^{-7}. \tag{1.38}$$

Therefore, for practical computation, the expressions to be evaluated are:

$$\begin{array}{ll} \sin\zeta - p^2 \sin 3\zeta + p^6 \sin 5\zeta & \text{[for } W \text{ and } W' \text{]} \\ \cos\zeta - 3p^2 \cos 3\zeta + 5p^6 \cos 5\zeta & \text{[for } W' \text{ only]} \end{array}$$

To reduce the number of evaluations of sines and cosines of complex argument, we make use of the summation algorithm [2], which in the present case takes the following form:

1. To evaluate $S_n = \sum_{m=0}^{n} a_m \sin(2m+1)\zeta$:

$$s = \sin\zeta \tag{1.39}$$

$$\alpha = 2 - 4s^2 \quad [= 2\cos 2\zeta] \tag{1.40}$$

$$u_n = a_n \tag{1.41}$$

$$u_{n-1} = a_{n-1} + \alpha a_n \tag{1.42}$$

$$\text{for } j = n-2\,(-1)\,0 \text{ do } u_j = a_j + \alpha u_{j+1} - u_{j+2} \tag{1.43}$$

$$S_n = (u_0 + u_1)s \tag{1.44}$$

2. To evaluate $C_n = \sum_{m=0}^{n} b_m \cos(2m+1)\zeta$:

$$c = \cos\zeta \tag{1.45}$$

$$\alpha = 4c^2 - 2 \quad [= 2\cos 2\zeta] \tag{1.46}$$

$$u_n = b_n \tag{1.47}$$

$$u_{n-1} = b_{n-1} + \alpha b_n \tag{1.48}$$

$$\text{for } j = n-2\,(-1)\,0 \text{ do } u_j = b_j + \alpha u_{j+1} - u_{j+2} \tag{1.49}$$

$$C_n = (u_0 - u_1)c \tag{1.50}$$

1.6 Thin wire in a circular tube (type D1)

By conformally mapping the circle $|z| = R$ representing a zero-potential cylinder onto the unit circle with the point z_0 representing a thin wire being mapped onto the origin, and then using a type A potential,

$$W(z) = -\frac{q}{2\pi\epsilon_0} \log \frac{(z - z_0)/R}{1 - z\bar{z}_0/R^2}$$

1.7 Ring of thin wires in a circular tube (type D2)

For a ring of n wires situated at $z_0 e^{i\nu 2\pi/n}$, $\nu = 0, 1, \cdots, n-1$ inside the cylinder $|z| = R$,

$$W(z) = -\frac{q}{2\pi\epsilon_0} \log \frac{(z^n - z_0^n)/R^n}{1 - (z\bar{z}_0/R^2)^n}$$

provided $z_0 \neq 0$, otherwise the D1 potential is used. The presence of two different potentials functions in the same cell makes that the capacitance matrix can be asymmetric.

1.8 Charges inside a polygon (type D3)

The potentials for wires inside a polygon are calculated by mapping the polygon onto the unit circle and then applying the potentials for wires inside a round tube (type D1). The mapping algorithm is described in the following paragraph.

1.9 Mapping a regular polygon onto the unit circle

(Contributed by G. A. Erskine)

From [4],

$$z = \frac{a}{\kappa} \int_0^w \frac{\mathrm{d}u}{(1 - u^n)^{2/n}}, \tag{1.51}$$

where

$$\kappa = \int_0^1 \frac{\mathrm{d}u}{(1 - u^n)^{2/n}} \tag{1.52}$$

$$= \frac{\Gamma(1 + 1/n)\Gamma(1 - 2/n)}{\Gamma(1 - 1/n)}. \tag{1.53}$$

On solving (1.51) by series reversion, we obtain

$$W(z) = \sum_{j=0}^{\infty} c_j \left(\frac{\kappa z}{a}\right)^{jn+1} \tag{1.54}$$

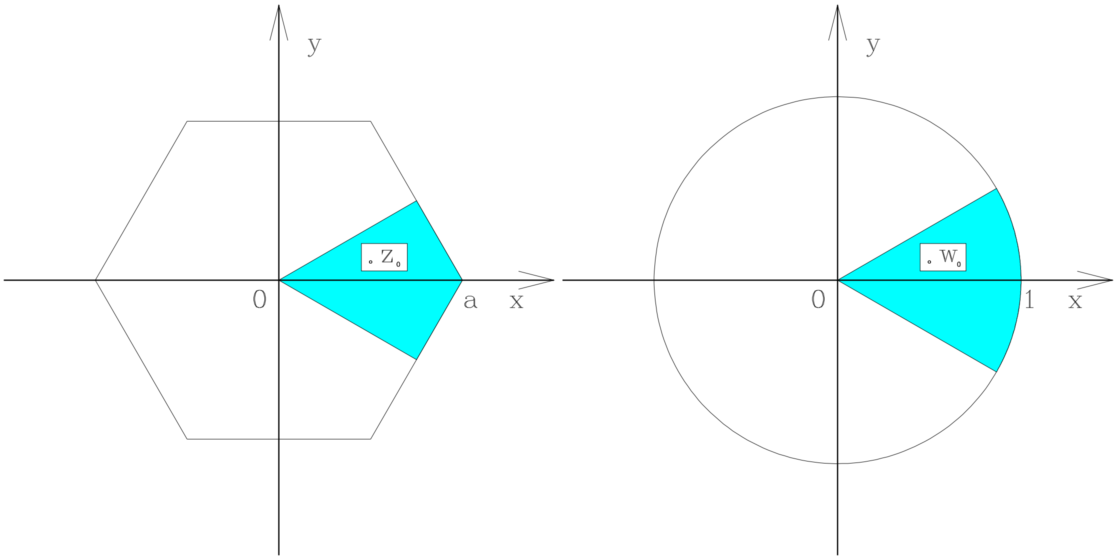

Figure 1.4: Mapping from the regular polygon with vertices at $ae^{i\nu 2\pi/n}$ onto the unit circle.

To obtain an expression which can be used in the neighbourhood of $z = a$ we set $u = 1 - v$ in (1.51), which yields

$$\kappa\left(1 - \frac{z}{a}\right) = \int_0^{1-w} \frac{\mathrm{d}v}{\left[1 - (1 - v)^n\right]^{2/n}}, \tag{1.55}$$

and hence, by reversion,

$$W(z) = 1 - \sum_{j=0}^{\infty} c'_j \left[\kappa(1 - z/a)\right]^{(j+1)\frac{n}{n+2}}. \tag{1.56}$$

1.10 The capacitance equations, boundary conditions

The equations to be solved to find the wire charges are known as capacitance equations, they are obtained by expressing the (known) potential of wire i in the (unknown) charges per unit length q_j on the wires $j = 1 \cdots n$.

The equipotential planes can be treated as if they were grounded if the linear potential V_{planes} generated by the planes alone is subtracted from all wire-potentials before the charges are calculated and added separately when the potential and electrostatic field are evaluated. The equipotential planes are assumed to be grounded in the rest of this chapter.

Explicit charge calculations for equipotential planes may be avoided if (multiple) mirror charges are introduced. See Fig 1.2 for their positions ($\odot$ = original wire and even mirror images, $\times$ = odd image).

The sum of all charges should always be zero (the energy of the electric field would be infinite). If there is at least one equipotential plane, the sum of all charges is automatically zero:

the charges and mirror charges cancel. The reference potential is set equal to zero in this case. The freedom to choose a reference potential can be exploited when planes are absent. To find the charges, we therefore have to solve the following equations:

$$V_i = \sum_{j=1}^{n_{\text{wires}}} C_{ij}^{-1} q_j + V_{\text{planes}}(z) + V_{\text{reference}} \tag{1.57}$$

$$\sum_{\text{wires + mirror wires}} q_j = 0 \tag{1.58}$$

where q_j are the charges to be found, and C^{-1} is the inverted capacitance matrix.

Making use of the expressions earlier on in this chapter, the elements of the capacitance matrix can be written as:

$$C_{ij}^{-1} = \begin{cases} \frac{-1}{2\pi\epsilon_0} \text{Re} \log F(z_i - z_j) & (i \neq j) \\ \frac{-1}{2\pi\epsilon_0} \text{Re} \log \left(d_i \lim_{z \to 0} \frac{F(z)}{z} \right) & (i = j) \end{cases}$$

and where F is a complex entire function such that

$$\text{Re} \log F(z) = \phi(z)$$

Once the charges per unit length are known, the potential at z can be evaluated from the formula:

$$V(z) = \sum_{j=1}^{n_{\text{wires}}} q_j \phi(z - z_j) + V_{\text{planes}}(z) + V_{\text{reference}}$$

1.11 Cylindrical geometry, internal coordinates

Cylindrical coordinates are convenient to describe chambers which contain planes that do not cross at right angles or which have 2 concentric tubes. Such cells are handled by conformally mapping the chamber to Cartesian coordinates and solving the fields using the recipes given earlier in this chapter. The coordinate mapping used by Garfield reads [6]:

$$(x, y, z) = (e^{\rho} \cos\phi, e^{\rho} \cos\phi, \zeta) \tag{1.59}$$

which translates circles into lines at constant ρ and radial lines into lines at constant ϕ. Circular and radial planes translate to planes at constant ρ and constant ϕ, while rotational symmetry becomes a ϕ periodicity.

The following points should however be noted:

- The origin in natural coordinates has no counterpart in (ρ, ϕ, ζ) coordinates. As a result, there should be no charges near the origin, and the field at the origin can not be computed. Configurations in which this is a limitation, can usually be handled using the tube potentials (Sections 1.6 and 1.7).

- Since ρ is not linear in r, with $r^2 = x^2 + y^2$, radially repetitive chambers can not be processed with this mapping.
- The mapping introduces a cut between $+\pi$ and $-\pi$. If the chamber does not have a declared ϕ-periodicity, then a 2π periodicity in ϕ is automatically added.

The transformation properties of some common geometrical objects can be derived using the transformation law (1.59):

scalars Scalars are, by definition, invariant.

Example: the electrostatic potential.

local vectors Local vectors behave like infinitesimal transformations, not like coordinates:

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = e^{\rho} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \mathrm{d}\rho \\ \mathrm{d}\phi \end{pmatrix}$$

Example: the drift velocity.

co-vectors Co-vectors are derivatives of a scalar. They transform according to:

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = e^{-\rho} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \partial_\rho \\ \partial_\phi \end{pmatrix}$$

In polar coordinates, the transformation reads:

$$\partial_{r\phi} = \partial_{\rho\phi}/r$$

Example: the components of the electrostatic field.

axial vectors We will call A an axial vector if

$$\forall V \text{ such that } V \text{ is a vector} : V \times A \text{ is a co-vector}$$

With this definition, the transformation of an axial vector reads:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = e^{2\rho} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_\zeta \end{pmatrix}$$

Example: the magnetic field.

Using these transformation properties, it follows that:

- The Laplace operator is the product of 2 gradients:

$$\nabla^2_{xy} = e^{-2\rho}\nabla^2_{\rho\phi}$$

The ρ-dependent scaling factor on the right hand side makes that extending the (x, y) coordinate system with a third coordinate that transforms as $\zeta = z$, leads to a violation of the Poisson equation under transformation. For simplicity, we will therefore limit the electric field to have only x and y components.

- The equation of motion in the absence of a magnetic field has the form of a proportionality between a vector and a co-vector:

$$v_{\mathrm{D}} = \mu(E)E$$

where $\mu(E)$ is called the mobility. As a rule, the mobility depends on the electric field strength and care has to be taken that the mobility is evaluated for the electric field strength in natural coordinates. In order to be able to use the same equation of motion in (x, y, z) and (ρ, ϕ, ζ), we define

$$\mu_{xyz} = e^{2\rho}\mu_{\rho\phi\zeta}$$

The above definition of the transformation property for the mobility is convenient also in the presence of a magnetic field since $|\mu B|$ transforms like a scalar. As a result, all terms in the equation of motion in the presence of a magnetic field have the same transformation behaviour, and the same equation can therefore be used with or without conformal mapping.

- Inner products of vectors and co-vectors are as usual scalars. The product of drift velocity, a vector, and the weighting field, a co-vector, is therefore a scalar. Thus, signals can be computed using the same equations whether using conformal mappings or not.

1.12 Zeros of the electric field

The points where $E = 0$ is satisfied are the natural counterparts of the singularities at the wire positions and as such play a key role in the understanding of the behaviour of drifting particles in the chamber. Wires (and their mirror images) are the end points of the drift-lines whereas zeros are bifurcation points in the drift-field. It follows that the drift-lines from these points delimit the various acceptance regions. It should be noted on the outset that limiting oneself to the no-B case is not a true constraint since the drift velocity vector is zero wherever E is zero, no matter the B field (see Section **??**).

1.12.1 The saddle shape of the zeros

E has a saddle point at its zeros owing to the harmonicity of the potential, as can be seen from a Taylor-expansion around the zero:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = -\begin{pmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \frac{\partial^2 V}{\partial y \partial x} & \frac{\partial^2 V}{\partial y^2} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} + \cdots$$

Assuming V satisfies the usual regularity conditions and using the harmonicity of V, one obtains after rotating over an angle

$$\tan \phi = \frac{\partial^2 V}{\partial x^2} / \frac{\partial^2 V}{\partial x \partial y}$$

a diagonal form:

$$\begin{pmatrix} E_u \\ E_v \end{pmatrix} = -\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} + \cdots$$

where λ is some (in the interesting case) non-zero number. The above treatment is valid for first order zeros, which are certainly the only ones one meets in practice; it would be interesting to investigate the existence of higher order zeros though.

The saddle shape can easily be inferred from this formula. An immediate -and important- consequence of this simple fact is that the argument of E (i.e. the angle of the E vector) changes by $-2pi$ over one full counter-clockwise loop around the saddle point. This is in marked contrast to wires where the argument changes by $+2pi$.

1.12.2 The principle of the argument

The principle of the argument is a convenient tool for counting the number of zeros and singularities (or poles) of a complex analytic function inside a given area. It simply states that for a closed loop γ and an meromorphic function f which has simple zeros and simple poles none of which lie on γ:

$$\Delta_\gamma \mathrm{Arg}\, f = 2\pi(\text{number of zeros} - \text{number of poles})$$

This is not the most general phrasing of the theorem but it is adequate for our purposes (see [7] for a proof). Recall that E (complex version) is *not* a meromorphic function but the complex conjugate of one that is. One merely has to change the sign of the change in argument to compensate for this. Hence, we find that:

$$\Delta_\gamma \mathrm{Arg}\, E = 2\pi(\text{number of wires} - \text{number of zeros})$$

1.12.3 Locating zeros

The principle of the argument can directly be applied to obtain via bisection regions that contain precisely one zero. The program uses a random search inside these areas to find good starting points and then steps towards the zeros with a first (!) order stepping method that assumes a saddle shape; higher order methods, no matter how sophisticated, are inevitably inefficient.

Bibliography

[1] H. Buchholz. *Elektrische und magnetische Potentialfelder*, chapter 3.11. Springer, 1957.

[2] C. W. Clenshaw. A note on the summation of chebyshev series. *Mathematical Tables & Aids to Computation [Journal name later changed to Mathematics of Computation]*, 9:118–120, 1955.

[3] E. Durand. *Magnétostatique*. Masson, Paris, 1968.

[4] Peter Henrici. *Applied and computational complex analysis*, volume 1 (Power Series - Integration - Conformal Mapping - Location of Zeros), chapter 5.12 III, page 411. John Wiley & Sons, 1986.

[5] I. A. Markushevich. *Theory of functions of a complex variable*, volume II, page 292. Prentice-Hall, 1965.

[6] Jerold E. Marsden. *Basic Complex Analysis*, chapter 5. W. H. Freeman and Company, 1973.

[7] Jerold E. Marsden. *Basic Complex Analysis*, chapter 6.2, pages 319–322. W. H. Freeman and Company, 1973.

[8] R. T. Whittaker and G.N. Watson. *A course of modern analysis*, chapter 21. Cambridge, 1927.

[9] R. T. Whittaker and G.N. Watson. *A course of modern analysis*, chapter 21.11, Example 3. Cambridge, 1927.

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