Analytic representations of Green's functions for numeric field computations near thin wires

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Numeric methods for electrostatic field computations meet difficulties in dealing with point charges or thin wires. Analytic representations of Green's functions in cylindrical coordinates are used as a workaround. Various (series and integral) representations for free space, outside a conducting wire, within a conducting tube and for a coaxial cable are given and their convergence properties are discussed. Closed expressions embracing the singularity combined with integral representations for the reaction field work best.

1 Introduction

A charge in a particle counter is subject to two fields: 1) The external static field provided by the electrodes; 2) the field due to other charges. A realistic computation taking into account the second contribution can be done only numerically. Such an approach meets some difficulties in dealing with point charges or thin wires as used in coaxial counters. These problems can be dealt with by using analytic expressions for the field of a point charge fulfilling the boundary conditions; so a Green's function is needed. No closed expressions are available. Unfortunately, expressions conprising infinite sums (and/or integrals) over particular solutions of the potential equation are slowly or not at all convergent if the field point is located at or near a certain surface passing through the source point. It is the main message of this paper that *difficulties with such series or integral representations occur on these surfaces although the field and the source point are different* so that the potential, the Green's function and the field components are finite, even analytic. We give modified

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representations, where this singularity is comprised in an analytic expression giving the Coulomb potential of a point charge in free space while the infinite sum of terms accounting for the boundary condition(s) has nice properties for numerical evaluations.

After discussing the singular surfaces specific to the Green's function in cylindrical coordinates ρ, ϕ, z (sect.2), the modification described above is performed for the Green's function outside an ideally conducting wire. In the next two sections we perform the same cure for a hollow tube and for a coaxial structure. We also show the problems besetting the eigenfunction expansions used for these two configurations. More details and figures can be found in a report, [1].

2 Green's function for free space

The Green's function for free space is the solution of the potential equation for a unit source, $\Delta G = -\delta(z-z')\,\delta(\phi-\phi')\,\delta(\rho-\rho')/\rho$, vanishing at infinity. The following integral representations of this function are quoted from [2]:

$$G_F(\rho, \phi, z; \rho', \phi', z') =$$

$$= \frac{1}{4\pi}\,\frac{1}{R} = \frac{1}{4\pi}\,\frac{1}{\sqrt{\rho^2 - 2\rho\rho'\cos(\phi-\phi') + \rho'^2 + (z-z')^2}}\,, \tag{1}$$

$$= \frac{1}{2\pi^2}\sum_{m=0}^{\infty} \epsilon_m \cos[m\ (\phi-\phi')] \int_0^{\infty} d\lambda\ \cos[\lambda(z-z')]\ I_m(\lambda\rho_<)K_m(\lambda\rho_>), \tag{2}$$

$$= \frac{1}{4\pi}\sum_{m=0}^{\infty} \epsilon_m \cos[m\ (\phi-\phi')] \int_0^{\infty} \mathrm{d}\zeta\ e^{-\zeta|z-z'|} J_m(\zeta\rho)J_m(\zeta\rho') \tag{3}$$

with $\rho_> = \max(\rho, \rho')$ and $\rho_< = \min(\rho, \rho')$. $\epsilon_m = (2 - \delta_{m0})$ is the Euler coefficient. The potential of a point charge Q located at ρ_0, ϕ_0, z_0 is $\Phi = Q\ G(\rho, \phi, z; \rho_0, \phi_0, z_0)/\varepsilon_0$; that due to a given charge distribution ρ_c is:

$$\Phi = \frac{1}{\varepsilon_0}\int \rho' d\rho' \int d\phi' \int dz'\ G(\rho, \phi, z; \rho', \phi', z')\ \rho_c(\rho', \varphi', z')\ . \tag{4}$$

The closed expression in eq.(1) is finite, even analytic, everywhere except if field and source point coincide. The two integral representations "remember" this singularity on surfaces passing throug the singular point. In fact, replacing the modified Bessel functions by their asymptotic expressions, [3], in the integrand of (2) displays the convergence of that integrand as:

$$\cos[\lambda(z-z')]\; e^{-\lambda(\rho_> - \rho_<)}/\lambda\sqrt{\rho_<\,\rho_>}. \tag{5}$$

It is seen that convergence is slow and conditional on the cylindrical surface $\rho_> = \rho_<$ and nonexistent if in addition $z = z'$ even if $\phi \neq \phi'$. The asymptotic behaviour of the integrand of (3) is:

$$e^{-\zeta|z-z'|}\,\Big[\cos[\zeta(\rho-\rho')] \;+\; \cos[\zeta(\rho+\rho') - m\pi]\Big]\Big/\zeta\sqrt{\rho\rho'}. \tag{6}$$

Convergence is slow and conditional in the plane $z = z'$ passing through the source point. The integrands for the field components E_ρ and E_z derived from that of the potential lack the λ or ζ in the denominator of (5) or (6); so they do not converge on the singular surfaces just described. It is obvious that all these deficiencies are very disadvantageous in numerical field computations based on such representations. In free space one will use the closed analytic expressions and will so avoid these difficulties. But in problems where conducting wires or tubes are present, the usual representations of the corresponding Green's functions display similar behaviour on and near the singular surfaces; this must and will be remedied below by use of the representations just discussed.

3 Green's function outside a wire

The electrostatic potential outside an ideally conducting wire of radius b and the corresponding Green's function G_E must fulfil the boundary condition: $\rho = b$ and/or $z = \pm\infty$: $\Phi = 0,\; G_E = 0$. The corresponding integral representation is found from formula (2) by adding partial solutions (regular for $\rho \geq b$) of the homogeneous potential equation such that the sum fulfils the boundary condition. This amounts to replacing the radial functions in the intgrands of (2) by the following ones:

$$[I_m(\lambda\rho_<)K_m(\lambda\rho_>) \;-\; K_m(\lambda\rho)K_m(\lambda\rho')I_m(\lambda b)/K_m(\lambda b)].$$

The first term of the new radial function is the same as above and has the disadvantageous properties discussed after (5). But the infinite sum comprising all these terms can be replaced with its closed expression (1). No closed expression is available for the infinite sum comprising all the second radial terms representing the reaction field due to the presence of the wire; it must be evaluated numerically. For that purpose it is advantageous to change the variable of integration from λ to λ/b. So the Green's function G_E is decomposed as:

$$G_E(\rho,\phi,z;\rho',\phi',z') \;=\; G_F(\rho,\phi,z;\rho',\phi',z') \;+ H_E(\rho,\phi,z;\rho',\phi',z') \tag{7}$$

with

$$H_E(\rho,\phi,z;\rho',\phi',z') \;=\; -\,\frac{1}{2\pi^2}\sum_{m=0}^{\infty}\epsilon_m \cos[m\ (\phi-\phi')]\ I_{Em}, \tag{8}$$

$$I_{Em} \;=\; \frac{1}{b}\int_0^{\infty} d\lambda\ \cos[\lambda(z-z')/b]\ K_m(\lambda\rho/b)K_m(\lambda\rho'/b)I_m(\lambda)/K_m(\lambda). \tag{9}$$

3.1 Numeric evaluation of the integrals

The numeric evaluation of the integrals (9) is not very difficult if some precautions are taken. The amplitudes of the integrands tend to zero rather fast with increasing values of the integration variable λ as long as at least one of the variables ρ and/or $\rho_0 > b$. In general, the integrands oscillate and, though the amplitude is very small, this property renders some numeric integration programs mad if the upper limit of integration is infinity. For the integrals we evaluated an upper limit $\lambda_u = 15$ gave amply sufficient accuracy; this was checked by an integration extending the upper limit to $\lambda_u = 30$.

For large λ the behaviour of the above integrand is:

$$\cos[\lambda(z-z')/b]\ e^{-\lambda[(\rho_>+\rho_<)/b-2]}/\lambda. \tag{10}$$

So convergence is good as long as not both ρ and ρ' are at or very near to b. So for the modified representation there remains only one difficult domain, if $2b\ \leq\ \rho+\rho'\ \leq 3b$; but the difficulty besetting the original expansion for any $\rho=\rho'$ is remedied.

As an example the charge distribution $\eta(\phi,z)=\varepsilon_0 E_\rho$ induced on the wire by a point charge Q is computed. So the derivative of eq.(7) w.r.t. ρ, thus also the derivative $I_{Em,\rho}$ of the integrals (9) are needed. These are shown in Figs.1 and 2, the surface charge density in Fig.3.

3.2 Use of the formulas in Garfield

The formulae derived above have been used in the program Garfield, [4].

4 Greens function within a conducting hollow tube

The electrostatic potential within an ideally conducting tube of radius a and the corresponding Green's function G_T must fulfil the boundary condition: $\rho = a$ and/or $z = \pm\infty$: $\Phi = 0$, $G_E = 0$. The corresponding integral representation is found from formula (2) by replacing the radial functions in the intgrands by the following ones:

$$[I_m(\lambda\rho_<)K_m(\lambda\rho_>) \; - \; I_m(\lambda\rho)I_m(\lambda\rho')K_m(\lambda a)/I_m(\lambda a)].$$

The infinite sum of integrals each containing the first term of the above backet is replaced with the closed expression in (1). In the remaining sum the variable of integration is changed from λ to λ/a. So the Green's function G_T is decomposed as:

$$G_T(\rho,\phi,z;\rho',\phi',z') \;=\; G_F(\rho,\phi,z;\rho',\phi',z') \; + H_T(\rho,\phi,z;\rho',\phi',z') \quad (11)$$

The second term is the same series as in (9) but with I_{Em} replaced by

$$I_{Tm} \;=\; \frac{1}{a}\int_0^\infty d\lambda \; \cos[\lambda(z-z')/a] \; I_m(\lambda\rho/a)I_m(\lambda\rho'/a)K_m(\lambda)/I_m(\lambda) \; . (12)$$

The behaviour for large λ of the integrand above is:

$$\cos[\lambda(z-z')/a] \; e^{-\lambda(2-\rho_> + \rho_<)/a]}/\lambda. \quad (13)$$

The integrals converge as long as $\rho+\rho' < 2a$. But convergence may be slow if both ρ and ρ' are at or very near to a. The problem occuring with $\rho = \rho' \; < \; 2a$ for any ρ in the original representation has been removed.

4.1 Eigenfunction expansion

As the radial domain is finite there is a discrete complete set of eigenfunctions. This may be used to represent the Green's function as [1], [5], [6] :

$$G_T(\rho,\phi,z;\rho',\phi',z') \;=\; \frac{1}{2\pi a}\sum_{m=0}^{\infty} \epsilon_m \cos[m\;(\phi-\phi')] \; K_m^T(\rho,\phi;\rho',\phi'), \quad (14)$$

$$K_m^T(\rho,\phi;\rho',\phi') \;=\; \sum_{n=1}^{\infty} e^{-j_{mn}|z-z'|/a} \; \frac{J_m(j_{mn}\rho/a)J_m(j_{mn}\rho'/a)}{j_{mn}\; J_m'^2(j_{mn})}. \quad (15)$$

j_{mn} are the zeros of the Bessel function J_m, i.e. $J_m(j_{mn}) = 0,\ n = 1, 2, ...$; they may be computed within an add-on package [8] of Mathematica. The series over n is conditionally and slowly convergent in the plane $z = z'$; the corresponding series for the field components E_ρ and E_z are not convergent. Similarly as in (6) this is shown [1], [9], [1] using asymptotic expressions for the Bessel functions and their zeros [3]. These slowly convergent or non-convergent series may be summed [9] by non-linear sequence transformations [10], but it appears more secure to use the integral representation (12). At the contrary, for large $|z/z'|$ these series converge rapidly, while wild oscillations due to $\cos[\lambda(z - z')]$ in the integrands of (9) or (12) may lead to trouble, though the total contribution will be small.

5 Greens function for a coaxial guide

5.1 *Integral representation*

The integral representation of the Green's function G_C valid in the annular domain $b \le \rho \le a$ and fullfilling the boundary conditions along the two ideally conducting electrodes and at infinity: $\rho = a,\ b$ and/or $z = \pm\infty:\ \Phi = 0,\ G_C = 0$ is obtained from the corresponding one for free space, eq.(2), by adding two solutions regular in this domain and subjecting the sum to the above boundary conditions. Thereafter it is decomposed as done before:

$$G_C(\rho, \phi, z; \rho', \phi', z') \;=\; G_F(\rho, \phi, z; \rho', \phi', z') \;+ H_C(\rho, \phi, z; \rho', \phi', z'). \quad (16)$$

The second term is the same series as in (9) but with I_{Em} replaced by

$$I_{Cm}(z - z', \rho, \rho') = \int_0^\infty d\lambda\ cos[\lambda(z - z')][I_m(\lambda\rho)c_m(\lambda) + K_m(\lambda\rho)d_m(\lambda)] \quad (17)$$

with

$$c_m(\lambda) \;=\; K_m(\lambda a)\ [I_m(\lambda b)K_m(\lambda\rho') \;-\; I_m(\lambda\rho')K_m(\lambda b)]/N_m(\lambda)\ , \quad (18)$$

$$d_m(\lambda) \;=\; I_m(\lambda b)\ [I_m(\lambda\rho')K_m(\lambda a) \;-\; I_m(\lambda a)K_m(\lambda\rho')]/N_m(\lambda)\ ; \quad (19)$$

$$N_m(\lambda) \;=\; I_m(\lambda b)K_m(\lambda a) - I_m(\lambda a)K_m(\lambda b). \quad (20)$$

Replacing the modified Bessel functions in the radial part of the integrand above by their asymptotic formulae gives after some rearrangements:

$$\frac{e^{-\lambda(2a\ -\ \rho\ -\ \rho')}\ +\ e^{-\lambda(\rho\ +\ \rho'\ -\ 2b)}\ -\ e^{-\lambda(2a\ -\ 2b\ +\ \rho\ -\ \rho')}\ -\ e^{-\lambda(2a\ -\ 2b\ -\ \rho\ +\ \rho')}}{2\lambda\sqrt{\rho\rho'}\ [1-e^{-2\lambda(a-b)}]}$$

The first two terms are decisive for the convergence of the integral; it converges unambiguously as long as both ρ and ρ' do not coincide with either the inner or outer radius.

In a coaxial counter the inner radius is very small as compared to the outer one, e.g. $a/b \approx 600$. In the integral above the variable λ is replaced with λ/b and it is rearranged such that the ratio $K_m(\lambda a/b)/I_m(\lambda a/b) \sim e^{-2\lambda a/b}$ gives the magnitude of terms. Since this ratio is very small these terms are small and what remains is the integrand of I_{Em}. These estimates justify the use of G_E in place of G_C for large values of the ratio a/b as long as ρ and ρ' are in the vicinity of the wire.

5.2 Eigenfunction expansion

The eigenfunction expansion of the Green's function G_C is given by formula (15) but with K_m^T replaced by ([1], [5], [6]):

$$K_m^C(\rho,\phi;\rho',\phi') \;=\; \frac{\pi^2}{4}\sum_{n=1}^{\infty} e^{-q_{mn}|z-z'|/a}\; q_{mn}\; \frac{R_{mn}(\rho)R_{mn}(\rho')}{J_m^2(q_{mn}b/a)/J_m^2(q_{mn})-1}. \quad (21)$$

The radial function defined as

$$R_{mn}(\rho) \;=\; J_m(q_{mn}\ \rho/a)Y_m(q_{mn}b/a) - Y_m(q_{mn}\ \rho/a)J_m(q_{mn}b/a).$$

fulfils the boundary condition at $\rho = b$ for any value of q_{mn}. But the boundary condition $\rho = a : R = 0$ is satisfied only for those q_{mn} which are roots of the transcendental equation $R_{mn}(\rho) = 0$. they may be computed within an add-on package [8] of Mathematica. However, for small values of the ratio b/a as it applies to coaxial counter geometries the numerical double precision procedure meets with difficulties for values of the azimuthal node number m larger than 6. But this is problem can be circumvented. For such values of m the Bessel function J_m is practically zero for small arguments; so it does not "feel" the central wire. Thus a radial function $J_m(j_{mn}\rho/a)$ may be used in place of $R_{mn}(\rho)$ and j_{mn} in place of q_{mn} for those values of n where difficulties appear; but be careful, for larger values of n $q_{mn}\rho/a$ may no longer be small enough.

The convergence properties of the infinite sum in (21) are the same as those of (15). The domain around $|z - z'| = 0$ is dangerous as discussed after eq.(15).

6 Conclusions

We have shown how potential and the field due to a point charge outside an ideally conducting wire, within a conducting tube or a coaxial cable may be repesented as a series of integrals which converge very well so that they are useful for numeric field computations.

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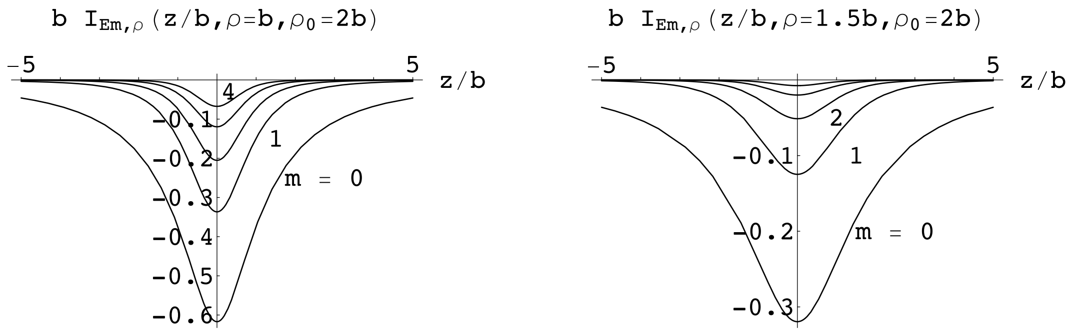

Fig. 1. The derivative w.r.t. ρ of the integrals $b\ I_{Em}$, $m = 0, ..., 4$, versus the relative longitudinal coordinate z/b for a point charge located at $\rho' = 2b,\ \phi' = z' = 0$ for two field point radii: $\rho = b$ (on the wire) and $\rho = 1.5b$ (halfway between wire and charge).

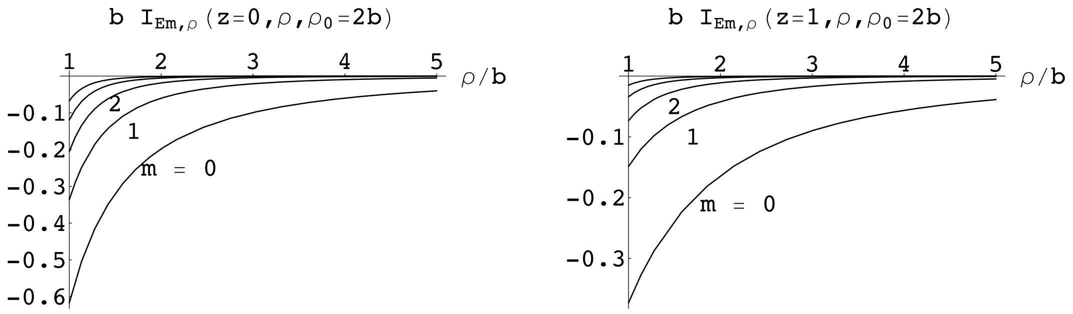

Fig. 2. The derivative w.r.t. ρ of the integrals $b\ I_{Em}$, $m = 0, ..., 4$, versus the radial field point coordinate ϕ for a point charge located at $\rho' = 2b,\ \phi' = z' = 0$ for two values of the longitudinal field point coordinate, $z = 0$ and $z = b$.

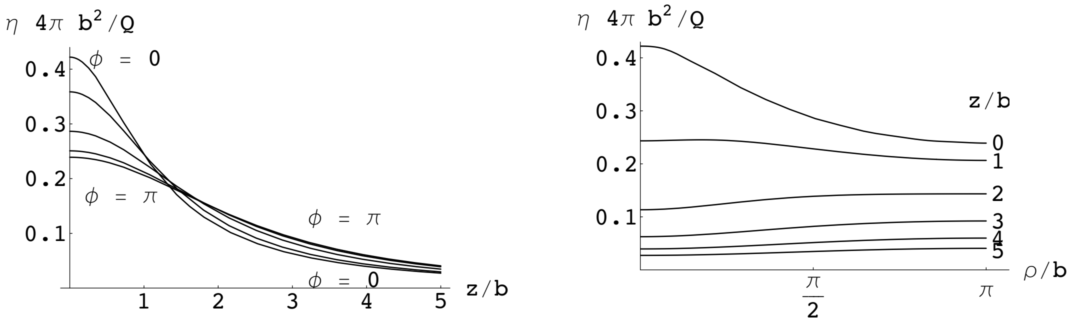

Fig. 3. The relative surface charge density $\eta(\phi, z)\ 4\pi b^2/Q$ on the wire (radius b) due to a point charge Q located at $\rho' = 2b,\ \phi' = z' = 0$ versus relative longitudinal coordinate z/b for angles $\phi = 0,\ \pi/4, ..., \pi$ (left) and versus azimuthal angle ϕ for $z/b = 0, 1, ..., 5$ (right).