Monte Carlo Generator for Synchrotron Radiation

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Abstract

This note describes a compact and efficient algorithm for the Monte Carlo generation of the synchrotron radiation photon spectrum. The minimal photon energy can be chosen between zero and about fifty times the critical energy.
The algorithm generates unweighted events, so that the number of photons per energy follows directly the synchrotron radiation spectrum.

1 Introduction

Synchrotron radiation is of major importance for e^+e^- storage rings. The properties of synchrotron radiation are well understood and described in standard textbooks [1]. The energy loss in synchrotron radiation of particle of charge e, velocity β, γ and energy E, travelling in a circular orbit of radius R is given[1] by :

$$\text{Energy loss/revolution} \qquad U = \frac{e^2}{3\epsilon_0 R}\beta^3\gamma^4 \tag{1}$$

For electrons we have $\gamma^4 = [E/(m_e c^2)]^4$ and $\beta \approx 1$:

$$U = \frac{e^2}{3\epsilon_0 R}\gamma^4 = 0.0884627 \ \text{MeV} \ \frac{(E/\text{GeV})^4}{R/\text{meter}} \tag{2}$$

The critical energy is usually[2] defined as $E_c = \frac{3}{2}\ \hbar c\gamma^3/R$, in agreement with reference [2], section III.40. With k we denote the photon energy E, expressed in units of the critical energy E_c:

$$k = \frac{E}{E_c} \tag{3}$$

Using this definition, the photon spectrum can be written independently of photon energy as

$$\frac{dn}{dk} = \sqrt{3}\,\alpha\,\gamma\ \cdot\ \mathbf{I(k)} \qquad \text{where} \qquad \mathbf{I(k)} = \int_k^\infty K_{5/3}(x)dx \tag{4}$$

[1]in SI units, the equivalent expression in c.g.s units as used in [2] is obtained formally by multiplying (1) with $4\pi\epsilon_0$

[2]Jackson in reference [1] instead uses E_c twice as large as defined here

where α is the fine-structure constant and $K_{5/3}$ a modified Bessel function of the third kind. For I(k), we use the efficient and precise Chebyshev series algorithm SYNRAD from H.H. Umstätter [3]. Useful formulas for the total number of photons can be found in the appendix. Synchrotron radiation can also be a serious source of background to the physics experiments and various equipment installed at storage rings. This necessitates the design of masks and collimators to absorb the synchrotron radiation.

Synchrotron radiation in the energy range of tenth of keV and above may traverse substantial amounts of material and undergo numerous scatterings. For a Monte Carlo simulation of synchrotron radiation and its passage through matter, it is convenient to start from a compact and efficient algorithm, that randomly generates photon energies following the synchrotron radiation spectrum (4).

2 Technique of Generation

The task is to find an algorithm, that effectively transforms the flat distribution given by standard pseudo-random generators into the desired distribution given by expression (4).

There are standard techniques to generate an arbitrary distribution. For references see [4] and [2], section III.37.

Generally, to obtain the probability distribution function f(k), the transformation can be constructed from the inverse of the cumulative distribution function $F(k) = \int_0^k f(x)dx$. The transformation function to be applied on the random numbers is $F^{-1}(l(k))$, where l(k) is a linear mapping of the form $a + bk$ to match the interval limits of the random numbers (usually the open interval from 0 to 1) with the ones of the desired distribution.
Monte Carlo generation of synchrotron radiation using standard numerical integration and interpolation techniques is discussed in [5].

The method presented here is based on simple, analytic approximations $f_a(k)$ for I(k), that are both integrable and invertible. The exact distribution is obtained by internal rejection, following a probability given by the ratio exact/approximate distribution.
In detail:
Approximations for the integral of the modified Bessel function I(k) can be derived from its low and high energy asymptotic limits:

$$\begin{aligned} \mathbf{I(k)} = \int_k^\infty K_{5/3}(x)dx &\approx a_1 \cdot k^{-2/3} \quad \text{for k} \ll 1 \\ &\approx a_2 \cdot e^{-k}/\sqrt{k} \quad \text{for k} \gg 1 \end{aligned}$$

We see, that I(k) is sufficiently well behaved for k $\rightarrow$ 0 such that the integral over I(k) evaluated from 0 to infinity is finite.
The high energy asymptotic limit is further simplified by dropping the term $\sqrt{k}$. The approximate frequency distributions are chosen as :

$$\begin{aligned} f_{a1}(k) &= a_1 \cdot k^{-2/3} \quad \text{for k} < 1 \\ f_{a2}(k) &= a_2 \cdot e^{-k} \quad \text{for k} \geq 1 \end{aligned}$$

The factors a_1,a_2 are fixed such, that the approximated functions, in their range of application, are always greater or equal than the exact function. In the limit k $\rightarrow$ 0, a_1 is also

known analytically : $a_1 = 2^{2/3}\Gamma(2/3)$
From integration and inversion of $f_{a1,2}$ we find the following transformations:

$$F_{a1}^{-1}(k) = \left[k_0^{1/3} + (1 - k_0^{1/3}) \cdot k\right]^3$$
$$F_{a2}^{-1}(k) = \max\,(1, k_0) - \log k \qquad (5)$$

The value k_0 allows to artificially cut off the photon spectrum at low energies. The complete spectrum is generated for $k_0 = 0$. For $k_0 > 1$ all photons are generated using approximation 2. Otherwise, both approximations are used and the fraction of photons generated from approximation 1 is obtained by analytic integration:

$$P = \frac{\int_{k_0}^{1} f_{a1}(x)dx}{\int_{k_0}^{1} f_{a1}(x)dx \;\; + \;\; \int_{1}^{\infty} f_{a2}(x)dx} \qquad (6)$$

Now, let $R_{1,2,3,4}$ be random numbers with a flat distribution in the open interval (0,1). The algorithm proceeds as follows.
1. step: if $R_1 < P$ use approximation 1, otherwise approximation 2.
2. step: generate a value for the photon energy $k = F_a^{-1}(R_2)$.
3. step: calculate the weight for the generated photon energy from the ratio exact over approximated probability : $w = f(k)/f_a(k)$.
4. step: if $R_4 < w$, keep the generated photon energy k as result, else restart at step 1 with four new random numbers.
A simple example might help in demonstrating that this way the generated photon energies follow exactly the photon spectrum given by equation (4):
If we obtain from step 2 a value of k for which the approximation is just 2 times the exact formula, then our internal weight as calculated in step 3 would be $w = f(k)/f_a(k) = 1/2$. Internally we have generated too many photons at energy k by a factor of two. Half of them would be rejected by step 4 so that we are left with the correct number of photons at energy k.
Figure 1 shows the distributions of weights as used internally in step 3 for $k_0 = 0$. The combined mean of the two weight distributions is 0.737 which means that only 26.3 % of the photons are rejected internally to go from the approximated to the precise frequency distribution.
The generation procedure is very fast (about 32 μsec per call on the CRAY XMP).

3 Use and Results

The procedure described in the previous section has been implemented in a FORTRAN function SYNGEN(k_0). The function value returned is the photon energy in units of the critical energy. The argument k_0 allows to restrict the generation to hard photons. To generate the full spectrum, the function should be called with the argument set to zero. To obtain only the extreme high energy tail, k_0 might be set to values as high as 50.

The results of 10^7 calls to SYNGEN with $k_0 = 0$ are shown in figure 2. The left histogram shows the photon spectrum and the right histogram the power spectrum.

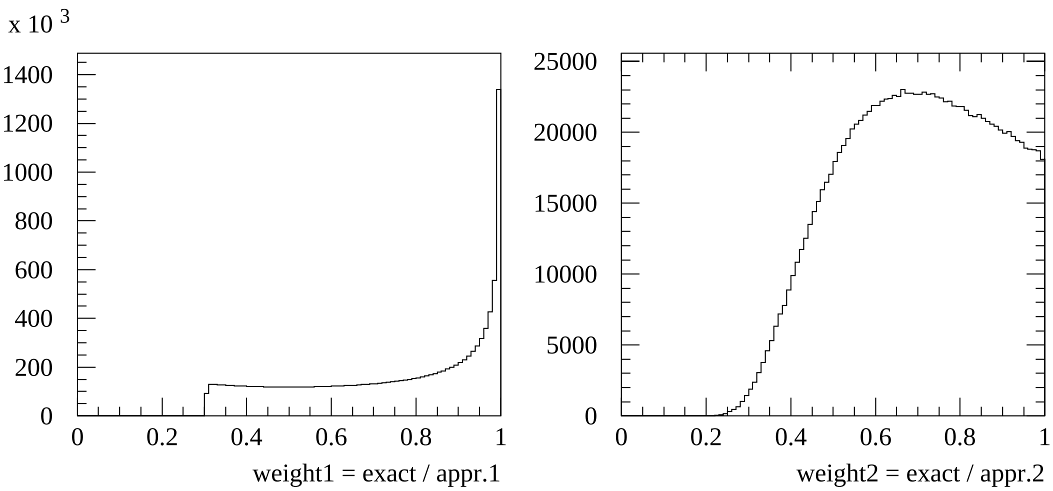

Figure 1: Weight distributions

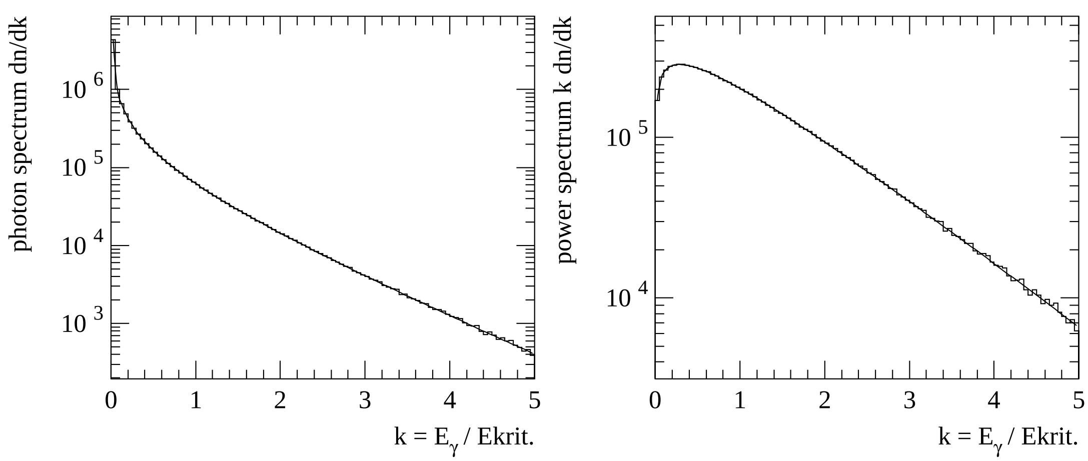

Figure 2: Generated spectra obtained from 10^7 calls to SYNGEN

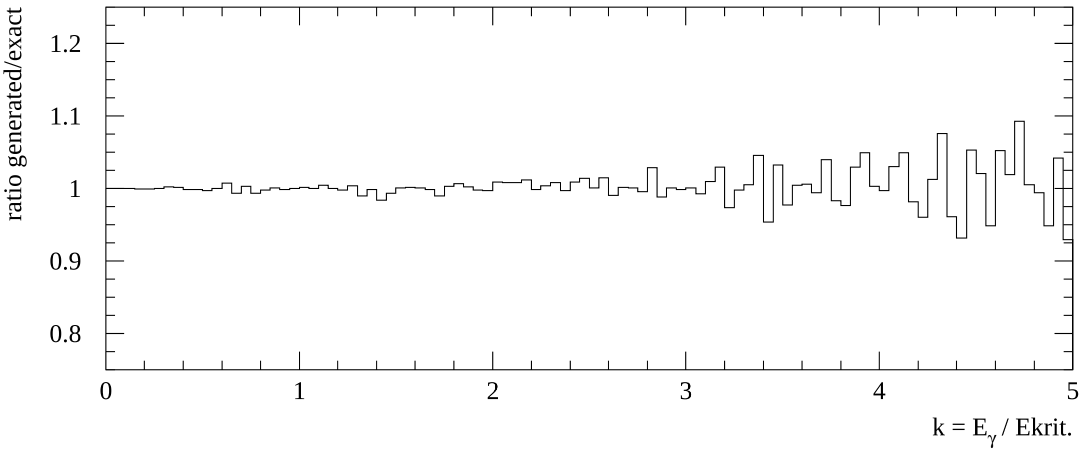

Figure 3: Ratio, generated/exact

for k above	photon spectrum	power spectrum
0.001	0.8772	0.9999
0.01	0.7381	0.9979
0.1	0.4628	0.9592
0.5	0.1896	0.7369
1.0	0.0868	0.5000
2.0	0.0233	0.2150
3.0	$0.7030 \cdot 10^{-2}$	$0.8864 \cdot 10^{-1}$
4.0	$0.2242 \cdot 10^{-2}$	$0.3571 \cdot 10^{-1}$
5.0	$0.7372 \cdot 10^{-3}$	$0.1417 \cdot 10^{-1}$
10.0	$0.3494 \cdot 10^{-5}$	$0.1243 \cdot 10^{-3}$
15.0	$0.1915 \cdot 10^{-7}$	$0.9930 \cdot 10^{-6}$
20.0	$0.1115 \cdot 10^{-9}$	$0.7593 \cdot 10^{-8}$
30.0	$0.4120 \cdot 10^{-14}$	$0.4146 \cdot 10^{-12}$
40.0	$0.1617 \cdot 10^{-18}$	$0.2152 \cdot 10^{-16}$
50.0	$0.6518 \cdot 10^{-23}$	$0.1079 \cdot 10^{-20}$

Table 1: Fraction of photons and photons weighted with their energy above a certain photon energy. The photon energy cut is given in units of the critical energy

On top of the histogram, the result of the exact spectrum has been drawn as smooth line. Figure 3 shows the ratio generated over exact. It is identical for the photon and power spectrum and shows perfect agreement, within statistical errors.

Table 1 gives results for the fractions above a certain photon energy. We see that only 8.68 % of the photons are radiated above the critical energy. The power spectrum instead, obtained from weighting each photon with its energy, is divided into equal halves by the critical energy, as expected.

4 Appendix: Some useful formulas for synchrotron radiation

To obtain the total number of photons or the total energy we use the relation [6]:

$$\int_0^\infty x^n \; K_\nu(x) \; dx = 2^{2n-1} \; \Gamma\left(\frac{1+n-\nu}{2}\right) \; \Gamma\left(\frac{1+n+\nu}{2}\right) \tag{7}$$

and the following property of the Γ - function:

$$\Gamma(x) \; \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

For the number of photons we evaluate :

$$\int_0^\infty I(k)dk = \int_0^\infty dk \int_k^\infty K_{5/3}(x)dx = \int_0^\infty xK_{5/3}(x)dx = \Gamma(1/6)\Gamma(11/6) = \frac{5\pi}{3} \tag{8}$$

and for the total energy :

$$\int_0^\infty k \cdot I(k)dk = \int_0^\infty kdk \int_k^\infty K_{5/3}(x)dx = \int_0^\infty \frac{x^2}{2} K_{5/3}(x)dx = \Gamma(2/3)\Gamma(7/3) = \frac{8\pi}{9\sqrt{3}} \tag{9}$$

From integration of (4) we obtain the total number of photons, radiated by one electron per turn:

$$n = \frac{5\pi\alpha\gamma}{\sqrt{3}} \tag{10}$$

The number of electrons, circulating per time interval dN/dt, follows from the beam current I and the electron charge e :

$$\frac{dN}{dt} = \frac{I}{e}$$

We divide by $l = 2\pi R$ to obtain the number of photons, radiated by an electron beam with current I, per unit length and time for a dipole with bending radius R as:

$$\frac{d^2\, n}{dl\; dt} = \frac{5\alpha}{2\sqrt{3}} \frac{I}{e} \frac{\gamma}{R} \;=\; 6.574 \cdot 10^{16} \, \frac{\gamma}{R} \, \frac{I/A}{s} \;=\; 1.2865 \cdot 10^{17} \, \frac{E/\mathrm{GeV}}{R} \, \frac{I/mA}{s} \tag{11}$$

For LEP I ($E_{beam} = 45.6$ GeV, R = 3096.175 m, $\gamma = 8.924 \cdot 10^4$) we find that

$$\frac{d^2\, n}{dl\; dt} = 1.895 \cdot 10^{15} / \text{ (m s mA)}$$

photons are radiated (per meter, second and milliamp of beam current) by the main bending magnets.

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