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Theory of linear and non-linear perturbations of betatron oscillations in alternating gradient synchrotrons

by

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1. Introduction

The stability criteria of orbits in particle accelerators are found by considering particles deviating slightly from the ideal orbit as to their energy, direction of velocity, or position. In the case of stability such particles carry out oscillations about the ideal orbit.

The oscillations are generally in three dimensions: two dimensions transverse, and one dimension longitudinal with respect to the orbit. The necessary restoring forces are provided by appropriately shaped guiding and focusing fields,and by an appropriate longitudinal position of the particles with respect to the phase of the accelerating field (considered as a travelling wave). Strictly speaking, the oscillations in all three dimensions are coupled. Usually, however, the "phase oscillations" with respect to the accelerating field are separated off, which in practical A.G.** synchrotrons, is justified by the fact that the phase oscillations have very much longer periods, than the "betatron oscillations" due to the focusing field. Thus the effect of the betatron oscillations on the phase oscillations averages out approximately, whereas the effect of the phase oscillations on the betatron oscillations consists of an adiabatic variation of parameters. It is, therefore, reasonable to disregard acceleration in an investigation of the fundamental stabilizing properties of the guiding and focusing field. The present report is restricted to this problem of pure betatron oscillations.

In the earlier stages of development, a linear approximation to the equations of motion of a deviating particle was used to obtain the stability criteria, which impose certain conditions on the guiding fields (Kerst and Server [1941]). After the invention of the A.G. focussing by Christofilos and Courant, Livingston and Snyder***, it was soon realized that very

* An outline of this report is contained in a paper given at the CERN Symposium held in Geneva in June 1956 (Hagedorn, Hine and Schoch [1956]).

** A.G. indicates abbreviation given throughout this report for "alternating gradient".

*** See Courant, Livingston and Snyder [1952] and Courant, Livingston, Snyder and Blewett [1953].

small imperfections of the guiding field of circular accelerators with A.G. focusing may cause the amplitude of oscillations about the stable orbit to grow indefinitely with time, thus upsetting the stability the perfect structure would provide*. This type of instability is due to resonance of the frequency of oscillation of the particle and the frequency of repetition of a perturbation in the guiding field, passed by the particle at each revolution.

In a later stage of development, the effects of non-linear terms in the equations of motion became important. Even in a synchrotron designed to make the orbits obey linear equations, there are non-linearities of kinematical origin, and there will usually be others due to unavoidable imperfections of the field. But, furthermore, the question has been raised whether the stability could eventually be improved by suitable artificial non-linearities.

It was known (Dennison and Berlin [1946], Courant [1949]) that non-linearities may give rise to new instabilities by excitation of "subresonances" *when the number* Q_1, Q_2 *of betatron oscillations in the radial and vertical* directions permit of integral combinations

$$n_1 Q_1 + n_2 Q_2 = p; \quad n_1, n_2, p \text{ integers}$$

(the resonances of "order" $|n_1| + |n_2| = 1$, and $|n_1| + |n_2| = 2$ are those appearing in linear oscillations). On the other hand, non-linearities make the betatron frequencies change with amplitudes and thus may prevent an indefinite build-up of amplitudes, as the resonance is shifted out of tune.

Whereas there are powerful and elegant methods of treating linear oscillations, and complete linear theories of stability of A.G. synchrotrons had been presented comparatively early on, mathematical methods are much less developed for non-linear oscillations. A great deal of exploration of non-linear effects on betatron oscillations has, therefore, been done by numerical computations**.

A more complete picture has at last been obtained by analytical perturbation methods, among which those presented by Moser [1955] and Sturrock [1955] formed the basis of further work at CERN, where Moser's method was

* first noticed by J.D. Lawson (see e.g. Bell [1953], Lüders [1953, 1955, 1956].

** see Adams and Hine [1953], Powell and Wright [1955] and unpublished calculations at Brookhaven and CERN.

extended to two dimensional oscillations and applied to betatron oscillations in circular accelerators by Hagedorn [1955, 1956]. The methods apply to basically linear systems with relatively small non-linear perturbations.

In the following a simpler approach is presented which, it is hoped, facilitates physical interpretation and application of the theory, expressed in terms of rather abstract formalisms in the afore-mentioned papers. The present approach covers linear perturbations at the same time and leads quickly to first approximation results (otherwise identical with those of the earlier papers).

It had been shown before (Schoch [1955]) that the results of Moser and Sturrock can be obtained in an elementary way in the case of a non-linear oscillator excited by periodically repeated kicks (such a system had been used before as a simplified model of an A.G. synchrotron with an imperfection in the early non-linear studies of Adams and Hine, quoted above). The method was to calculate the change of amplitude and phase of the oscillation brought about by the perturbing kicks, making use of the fact that those quantities, which would be constant for the unperturbed motion, vary only slowly if the perturbations are small. This method of "slowly varying amplitude and phase" is adapted in the following to the case of an A.G. synchrotron with quite general perturbations. The procedure partly follows a paper by Beth [1910] who treated non-linear coupling resonances in systems with several degrees of freedom long ago. *For the sake of simple presentation the theory of one*-dimensional oscillations will be given in detail first, and extended to two dimensional oscillations in the last sections. The more satisfactory part of the theory given is restricted to synchrotrons whose parameters are rigidly constant in time. The "dynamic" behaviour under conditions of varying parameters (e.g. due to phase oscillations) is known to be appreciably different in certain cases. It is considered to some extent in one of the last sections.

The conclusions of the theory regarding requirements on synchrotron parameters and construction tolerances are summarized in the last section with *a view to the CERN proton synchrotron (CERN PS) in particular.*

Details necessary for the analysis of a practical synchrotron design are collected in a voluminous appendix.

It must be emphasized that already in 1950, Judd [1950] had treated the problem of non-linear oscillations in betatrons and synchrotrons using the method of slowly varying amplitude and phase in the version of Krylov and Bogoliubov [1937]. His work contains all of the fundamental results having been given later in the papers mentioned above. Unfortunately it did not become known at CERN before 1956, and seems to have passed unnoticed by most of the accelerator centers, as it has not appeared in print.

Further work which has to be mentioned was done partly along similar lines, by Cole [1954], Symon [1954], Courant [1956] and Kolomenski [1956] and more recently by Parzen [1956, 1957] who used an entirely different approach.

2. Method of "variation of canonical constants"

As pointed out in the introduction, we try to represent the oscillations of the actual non-linear system by letting amplitude and phase of the *oscillations of the unperturbed linear system undergo slow variations. That is,* we try to solve the non-linear equations by "variation of constants", as amplitude and (inital) phase are constants of the motion of the unperturbed linear system. If

$$H(p,q,t) = H^{(o)}(p,q,t) + H^{(1)}(p,q,t) \tag{2.1}$$

is the Hamiltonian of the non-linear system, $H^{(o)}$ and $H^{(1)}$ being the Hamiltonians of the unperturbed system and the perturbation, and if

$$p = f(t,a,b)$$
$$q = g(t,a,b)$$

is the motion of the unperturbed system, with a, b integration constants, we solve the complete system by considering a, b as functions of t :

$$p = f(t, a[t], b[t])$$
$$q = g(t, a[t], b[t])$$

Introducing this into the Hamiltonian equations of motion :

$$\dot{p} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial a}\dot{a} + \frac{\partial f}{\partial b}\dot{b} = -\frac{\partial H}{\partial q}$$

$$\dot{q} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial a}\dot{a} + \frac{\partial g}{\partial b}\dot{b} = \frac{\partial H}{\partial p}$$

and solving for $\dot{a}$, $\dot{b}$ we obtain

$$\dot{a} = \frac{-\left(\frac{\partial H}{\partial p} - \frac{\partial g}{\partial t}\right)\frac{\partial f}{\partial b} - \left(\frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial b}}{J}$$

$$\dot{b} = \frac{\left(\frac{\partial H}{\partial p} - \frac{\partial g}{\partial t}\right) \frac{\partial f}{\partial a} + \left(\frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}\right) \frac{\partial g}{\partial a}}{J}$$

where

$$J = \begin{vmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial b} \end{vmatrix} \qquad (2.3)$$

Using now the fact that

$$\frac{\partial f}{\partial t} = - \frac{\partial H^{(o)}}{\partial q}$$

$$\frac{\partial g}{\partial t} = \frac{\partial H^{(o)}}{\partial p}$$

the equations for a, b take the form

$$\dot{a} = \frac{-\left(\frac{\partial H^{(1)}}{\partial p} \frac{\partial f}{\partial b} + \frac{\partial H^{(1)}}{\partial q} \frac{\partial g}{\partial b}\right)}{J} = - \frac{1}{J} \frac{\partial H^{(1)}[a, b, t]}{\partial b}$$

$$\dot{b} = \frac{\left(\frac{\partial H^{(1)}}{\partial p} \frac{\partial f}{\partial a} + \frac{\partial H^{(1)}}{\partial q} \frac{\partial g}{\partial a}\right)}{J} = \frac{1}{J} \frac{\partial H^{(1)}[a, b, t]}{\partial a} \qquad (2.4)$$

Thus if $J = 1$, a, b simply obey equations derived from the perturbation Hamiltonian $H^{(1)}$, after substitution of p, q by a, b by means of the solution for the unperturbed system. In particular, $J = 1$ is satisfied if in (2.2) such constants as form a set of canonical variables are chosen (i.e. if (2.2) represents a canonical transformation from p, q to a, b).

3. Introduction of amplitude and phase as variables

Considering only the radial oscillations in the plane of symmetry of an A.G. synchrotron (the two dimensional case will be treated in section

12), the equations of motion for the radial displacement x may be written (see Appendix I) :

$$\frac{dx}{d\vartheta} = \frac{\partial}{\partial x'}(H^{(0)} + H^{(1)}) = x' + \frac{\partial H^{(1)}}{\partial x'}$$

$$\frac{dx'}{d\vartheta} = - \frac{\partial}{\partial x} (H^{(0)} + H^{(1)}) = n(\vartheta)x - \frac{\partial H^{(1)}}{\partial x} \tag{3.1}$$

where ϑ is the angular position on the circumference, x' the "canonical momentum" (defined by the first of the equations) $n(\vartheta)$ the field index with period $\Theta = \frac{2\pi}{M}$ if there are M magnet periods. The part $H^{(1)}(x, x')$ of the Hamiltonian contains all linear and non-linear perturbation terms whereas

$$H^{(0)} = \frac{(x')^2 - n(\vartheta)\ x^2}{2}$$

defines the unperturbed system, in which $x' = \frac{dx}{d\vartheta}$, and which obeys the equation :

$$\frac{d^2x}{d\vartheta^2} - n(\vartheta)\ x = 0.$$

Its solution can be written in the well known Floquet form

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{a} \left[\begin{pmatrix} w_1(\vartheta) \\ w_2(\vartheta) \end{pmatrix} e^{i(Q\vartheta + \varphi)} + \begin{pmatrix} w_1^*(\vartheta) \\ w_2^*(\vartheta) \end{pmatrix} e^{-i(Q\vartheta + \varphi)} \right] \tag{3.2}$$

where a and φ are constants of the motion, and Q follows from the characteristic equation for the transfer matrix T_Θ over one period Θ of $n(\vartheta)$:

$$T_\Theta \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = e^{iQ\Theta} \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$$

$$T_\Theta \begin{pmatrix} w_1^*(0) \\ w_2^*(0) \end{pmatrix} = e^{-iQ\Theta} \begin{pmatrix} w_1^*(0) \\ w_2^*(0) \end{pmatrix} \tag{3.3}$$

The Floquet factors $\begin{pmatrix} w_1(\vartheta) \\ w_2(\vartheta) \end{pmatrix}$, which are periodic with period Θ, are related to the eigenvectors $\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$ in the following way

$$\begin{pmatrix} w_1(\vartheta) \\ w_2(\vartheta) \end{pmatrix} = e^{-iQ\vartheta} T_\vartheta \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}, \quad 0 < \vartheta < \Theta, \tag{3.4}$$

where T_ϑ is the transfer matrix connecting the points ϑ and 0.

We are going to consider (3.2) as a transformation of x, x' to new variables φ, a. In order that the Jacobian of this transformation be equal to 1, the Floquet factors have to be normalized properly :

$$J = \begin{vmatrix} \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial a} \\ \frac{\partial x'}{\partial \varphi} & \frac{\partial x'}{\partial a} \end{vmatrix} = \begin{vmatrix} i\sqrt{a}\left(w_1 e^{i\alpha} - \overset{*}{w}_1 e^{-i\alpha}\right) & \frac{1}{2\sqrt{a}}\left(w_1 e^{i\alpha} + \overset{*}{w}_1 e^{-i\alpha}\right) \\ i\sqrt{a}\left(w_2 e^{i\alpha} - \overset{*}{w}_2 e^{-i\alpha}\right) & \frac{1}{2\sqrt{a}}\left(w_2 e^{i\alpha} + \overset{*}{w}_2 e^{-i\alpha}\right) \end{vmatrix}$$

$$= i \begin{vmatrix} w_1 & \overset{*}{w}_1 \\ w_2 & \overset{*}{w}_2 \end{vmatrix} = i\,|T_\vartheta| \begin{vmatrix} w_1(0) & \overset{*}{w}_1(0) \\ w_2(0) & \overset{*}{w}_2(0) \end{vmatrix} = i \begin{vmatrix} w_1(0) & \overset{*}{w}_1(0) \\ w_2(0) & \overset{*}{w}_2(0) \end{vmatrix} = 1$$

where $\alpha = Q\vartheta + \varphi$.

The eigenvector $\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$ follows from (3.3) :

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = C \begin{pmatrix} T_{12} \\ e^{iQ\Theta} - T_{11} \end{pmatrix} = C \begin{pmatrix} T_{12} \\ \frac{T_{22} - T_{11}}{2} + i\sin Q\Theta \end{pmatrix}, \tag{3.5}$$

T_{11}, T_{12}, T_{22} being elements of the transfer matrix T_Θ and C a normalization factor, to be chosen such that $J = 1$:

$$J = C\overset{*}{C} \cdot 2T_{12}\sin Q\Theta = 1$$

$$C = (2T_{12}\sin Q\Theta)^{-\frac{1}{2}} \quad . \tag{3.6}$$

The choice of a (the square of the amplitude) rather than $\sqrt{a}$ (the amplitude) as a second variable is imposed by the preservation of phase space volume in a transformation with Jacobian $J = 1$.

Expanding the Floquet factors in a Fourier series with fundamental period ϑ, we may write for $w_1(\vartheta) \equiv w(\vartheta)$ (omitting the suffix as only $w_1(\vartheta)$ will be needed later on) :

$$w(\vartheta) = \sum_{\nu=-\infty}^{+\infty} w_{\nu M}\, e^{-i\nu M\vartheta},$$

$M = \frac{2\pi}{\Theta}$ being the number of magnet periods on the circumference 2π.

Fourier expansions of $w(\vartheta)$ for structures of practical interest are given in Appendix IV. Choosing the n-value of a pure square wave structure (alternating gradient without field-free sections) such as to make the essential parameters equal to those of the CERN A.G. synchrotron, i.e.

$$Q\Theta = \frac{\pi}{4}, \quad (Q = 6.25), \quad \Theta = \frac{2\pi}{M} = \frac{2\pi}{50},$$

the following numerical values are obtained for n and the Fourier coefficients of the Floquet factor $w(\vartheta)$ normalized according to (3.6)) :

$$n = 336.5$$

$$w_0 = 0.289$$

$$\frac{w_M}{w_0} = 0.1122 \qquad \frac{w_{-M}}{w_0} = 0.0678$$

$$\frac{w_{2M}}{w_0} = 0.00207 \qquad \frac{w_{-2M}}{w_0} = 0.00050$$

....

These figures show to what extent, under conditions of practical interest, the transverse oscillation in an A.G. synchrotron is almost sinusoidal. The wriggle due to the A.G. structure amounts to about 15 percent of the amplitude of the "smooth motion" and is itself almost sinusoidal.

Frequently it is useful to compare the smooth part of the motion in the A.G. synchrotron with the oscillation of a simple harmonic oscillator of natural frequency Q. Considering the unperturbed linear system as being a harmonic oscillator, the Floquet factor in (3.2) becomes a constant vector

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2Q}} \\ i\sqrt{\frac{Q}{2}} \end{pmatrix}.$$

With the above numerical data ($Q = 6.25$), $w_1 = w = 0.283$, which is very nearly equal to the Fourier coefficient $w_o = 0.289$ giving the smooth motion in the foregoing example of an A.G. synchrotron.

4. Equations of motion in the φ,a representation

According to section 2), these are derived from the perturbation Hamiltonian $H^{(1)}$, after having expressed x, x' by φ,a by means of (3.2)

$$\frac{d\varphi}{d\vartheta} = \frac{\partial H^{(1)}(\varphi,a,\vartheta)}{\partial a}$$
$$\frac{da}{d\vartheta} = -\frac{\partial H^{(1)}(\varphi,a,\vartheta)}{\partial \varphi} \qquad (4.1)$$

Referring to Appendix I, $H^{(1)}$ consists of potential energy only, if we can disregard non-linear terms of kinematical origin. This is justified unless the non-linearities of the guiding field are extremely small. For purely radial oscillations $H^{(1)}$ then takes the form

$$H^{(1)} = V(x,\vartheta) = \sum_{k=o}^{\infty} V_k(\vartheta)\, x^k, \qquad (4.2)$$

the $V_k(\vartheta)$ being in general periodic functions of ϑ, of period Θ for a perfect, of period 2π for an imperfect synchrotron. Substitution of φ,a for x gives

$$H^{(1)} = \sum V_k(\vartheta)x^k = \sum_k \sum_{\substack{\ell,m>o \\ \ell+m=k}} \binom{\ell+m}{m} V_{\ell+m}(\vartheta)w(\vartheta)^{\ell}\, \overset{*}{w}{}^m(\vartheta)\,a^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta+\varphi)}$$

$$= \sum_k \sum_{\ell+m=k} v_{\ell m}(\vartheta)\,a^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta+\varphi)} \qquad (4.3)$$

$$= \sum_k \sum_{\ell+m=k} \sum_q v_{\ell mq}\, a^{\frac{\ell+m}{2}}\, e^{i[(\ell-m)(Q\vartheta+\varphi) - q\vartheta]}$$

where $v_{\ell mq}$ *are Fourier coefficients defined by*

$$\binom{\ell+m}{m} v_{\ell+m}(\vartheta)\, w^{\ell}(\vartheta)\overset{*}{w}{}^{m}(\vartheta) = v_{\ell m}(\vartheta) = \sum^{+\infty} v_{\ell mq} e^{-iq\vartheta} \tag{4.4}$$

The summations are over $\ell \geq 0$, $m \geq 0$, $\ell + m = k$. We note that

$$\overset{*}{v}_{\ell mq} = v_{m\ell(-q)}. \tag{4.5}$$

We further note that $H^{(1)}$ would have the same general form (4.3) also if terms containing the canonical momentum x' would occur in the original $H^{(1)}$.

5. Approximate equations for the "slowly varying" parts of amplitude and phase

The Hamiltonian (4.3) is composed of terms whose explicit dependence on ϑ is of oscillatory character, the frequencies being $[(\ell-m)Q - q]$. We expect - for relatively small perturbations - the amplitude and phase to change only little within the period of one oscillation or even one revolution (several oscillations). We may, therefore, expect to obtain approximate equations of motion by keeping only those terms in the Hamiltonian the frequency of which is either zero or very small

$$|(\ell-m)Q-q| = 0 \quad or \quad << 1. \tag{5.1}$$

These terms are (i) those belonging to $\ell-m = o$, $q = o$, and (ii) those for which the wave number Q satisfies, or almost satisfies, a relation

$$nQ - p = o, \quad n,p \text{ integers.} \tag{5.2}$$

The terms obeying a relation (5.2) are the ones exciting subresonances, as will be discussed later.

That part of the Hamiltonian H(index $^{(1)}$ omitted from here on) which is composed only of zero and low frequency terms defined by (5.1) will be denoted by $\bar{H}$.

The approximation may be put on a more rigorous basis and improved in the following way : Decomposing the motion in the form

$$\begin{aligned} \varphi &= \phi(\vartheta) + \tilde{\varphi}\,(\phi, A, \vartheta) \\ a &= A(\vartheta) + \tilde{a}\,(\phi, A, \vartheta) \end{aligned} \tag{5.3}$$

where ϕ,A represent the slowly varying part of the motion, and the rapidly oscillating parts $\tilde{\varphi}$, $\tilde{a}$ have to be added to give the exact motion, (5.3) can be considered as a canonical transformation to new variables ϕ, A, defined by the requirement that the new Hamiltonian contain only zero or low frequency terms in the sense of (5.1). Proceeding in this way it will be seen in the next section that the transformed Hamiltonian is - in a first approximation - just the $\bar{H}$, obtained by omitting the high frequency terms in the original perturbation Hamiltonian H.

It will furthermore turn out that the higher approximations do not affect the general form of the transformed Hamiltonian, and that the first approximation is sufficient for most of the problems concerned with small (linear and non-linear) perturbations. *One may, therefore, often ignore the* more rigorous reasoning given in the next section, as the first approximation seems immediately plausible.

The transformation to dynamical variables in which the Hamiltonian is aslittle as possible dependent on ϑ is the essential feature of the procedure used by Moser [1955]. His new variables arrived at may, therefore, be interpreted as the slowly varying part of the motion in the sense we have been explaining above. Obviously this interpretation is useful as long as the residual rapid motion $\tilde{\varphi}$, $\tilde{a}$ remains relatively small, that is, if the above *transformation is almost infinitesimal. The conditions for this being the* case will be discussed in section 7.

6. Systematic procedure for obtaining the slowly varying parts of phase and amplitude.

Following the recipe of analytical dynamics we represent the canonical transformation (5.3) from φ, a to ϕ, A by means of a generating function

$$S(\varphi,A,\vartheta) = \varphi A + \sigma(\varphi,A,\vartheta)$$

where upon

$$\phi = \frac{\partial S}{\partial A} = \varphi + \frac{\partial \sigma}{\partial A}$$
$$a = \frac{\partial S}{\partial \varphi} = A + \frac{\partial \sigma}{\partial \varphi} \qquad (6.1)$$

and the new Hamiltonian

$$\bar{H} = H + \frac{\partial S}{\partial \vartheta} = \sum_{k} \sum_{\ell+m=k} v_{\ell m}(\vartheta)\, a^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta+\varphi)} + \frac{\partial \sigma}{\partial \vartheta} \tag{6.2}$$

Putting

$$\bar{H} = \sum_{k} \sum_{\ell+m=k} h_{\ell m}(\vartheta) A^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta + \phi)} , \; . \tag{6.3}$$

with coefficients $h_{\ell m}(\vartheta)$ to be determined later, and assuming σ in a similar form

$$\sigma = \sum_{k} \sum_{\ell+m=k} \sigma_{\ell m}(\vartheta) A^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta + \varphi)} \tag{6.4}$$

we find from (6.1) and (6.2) that $h_{\ell m}$ and $\sigma_{\ell m}$ have to satisfy the following equation :

$$\sum\sum\left[\frac{d\sigma_{\ell m}}{d\vartheta} + i(\ell-m)Q\sigma_{\ell m}\right] A^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta + \varphi)} =$$

$$\sum\sum\left\{ h_{\ell' m'} A^{\frac{\ell'+m'}{2}} e^{i(\ell'-m')(Q\vartheta+\varphi+\frac{\partial\sigma}{\partial A})} - v_{\ell'' m''}\left(A + \frac{\partial\sigma}{\partial\varphi}\right)^{\frac{\ell''+m''}{2}} e^{i(\ell''-m'')(Q\vartheta + \varphi)}\right\}$$

Introducing (6.4) on the r.h.s. and using power expansions with respect to σ, this becomes

$$\sum\sum\left[\frac{d\sigma_{\ell m}}{d\vartheta} + i(\ell-m)Q\sigma_{\ell m}\right] A^{\frac{\ell+m}{2}} e^{i(\ell-m)(Q\vartheta+\varphi)} =$$

$$\sum\sum h_{\ell' m'} \left[1+i(\ell'-m') \sum\sum \sigma_{\ell'' m''} \frac{\ell''+m''}{2} A^{\frac{\ell''+m''}{2}-1} e^{i(\ell''-m'')(Q\vartheta+\varphi)} + \dots\right] \times$$

$$A^{\frac{\ell'+m'}{2}} e^{i(\ell'-m')(Q\vartheta + \varphi)}$$

$$- \sum\sum v_{\ell'' m''}\left[A^{\frac{\ell''+m''}{2}} + \frac{\ell''+m''}{2} \sum\sum i(\ell'-m')\sigma_{\ell' m'} A^{\frac{\ell''+m''+\ell'+m'}{2}-1} e^{i(\ell'-m')(Q\vartheta+\varphi)} + \dots\right] \times$$

$$e^{i(\ell''-m'')(Q\vartheta+\varphi)} =$$

(continued on next page)

$$\sum\sum (h_{\ell m} - v_{\ell m})\, A^{\frac{\ell+m}{2}}\, e^{i(\ell-m)(Q\vartheta+\varphi)}$$

$$+ \sum\sum\sum\sum i^{\frac{(\ell'-m')(\ell''+m'')}{2}} (h_{\ell' m'}\sigma_{\ell'' m''} - v_{\ell'' m''}\sigma_{\ell' m'}) A^{\frac{\ell'+m'+\ell''+m''-2}{2}}$$

$$e^{i(\ell'-m'+\ell''-m'')(Q\vartheta+\varphi)} + \dots \Big]$$

where we have written down the expansions only up to the first order in σ.

This equation is satisfied when the $\sigma_{\ell m}$ singly satisfy the equations :

$$\frac{d\sigma_{\ell m}}{d\vartheta} + i(\ell-m)Q\sigma_{\ell m} = h_{\ell m} - v_{\ell m} + \sum_{\substack{\ell'+m'+\ell''+m'' = \ell+m+2 \\ \ell'-m'+\ell''-m'' = \ell-m}} i^{\frac{(\ell'-m')(\ell''+m'')}{2}} (h_{\ell' m'}\sigma_{\ell'' m''} - v_{\ell'' m''}\sigma_{\ell' m'}) + \dots \tag{6.5}$$

Requiring now that $h_{\ell m}$, $\sigma_{\ell m}$ be periodic, like $v_{\ell m}$, it is convenient to use Fourier expansions

$$\sigma_{\ell m}(\vartheta) = \sum_q \sigma_{\ell m q}\, e^{-\ell q\vartheta}, \quad \text{etc.}$$

The equations (6.5) then are equivalent to the following equations for the Fourier component

$$\sigma_{\ell m q} = \frac{h_{\ell m q} - v_{\ell m q} + \sum_{q'} \sum i^{\frac{(\ell'-m')(\ell''+m'')}{2}} h_{\ell' m' q'}\sigma_{\ell'' m'', q-q'} - v_{\ell'' m'' q'}\sigma_{\ell' m', q-q'} + \dots}{i[(\ell - m)\, Q - q]} \tag{6.6}$$

the summation on the r.h.s. to be carried out over the combinations of $\ell' m' \ell'' m''$ indicated in (6.5).

In principle, one could determine transformation coefficients $\sigma_{\ell m q}$ from these equations for any choice of the coefficients $h_{\ell m q}$ of the new Hamiltonian (subject to conditions of solubility, of course). In accordance with the programme of section 5, we require all $h_{\ell m q}$ to vanish except those of the zero or low frequency terms for which either

$$\ell - m = o, \quad q = o$$

or a subresonance condition $nQ - p \approx 0$, holds,

$$\ell - m = n, \quad q = p.$$

(In fact, due to the denominator in (6.6) the $h_{\frac{k}{2}\frac{k}{2}o}$ and $h_{\frac{k+n}{2}\frac{k-n}{2}p}$ are not allowed to vanish generally, otherwise the transformation coefficients would become infinite or very large). The appropriate choice for the remaining coefficients $h_{\frac{k}{2}\frac{k}{2}o}$ and $h_{\frac{k+n}{2}\frac{k-n}{2}p}$ follows from (6.6) by requiring finite solutions $\sigma_{\ell mq}$. Solving these equations by iteration* and using the required properties of $h_{\ell mq}$, we obtain in first approximation.

$$\sigma_{\ell mq} = \frac{-v_{\ell mq}}{i\;[(\ell-m)Q-q]} \quad , \quad \text{for} \quad (\ell-m)Q-q \not\approx 0. \tag{6.7a}$$

For $(\ell-m)Q-q \approx 0$, we have to put in first approximation

$$\left[\begin{aligned} h_{\frac{k}{2}\frac{k}{2}o} &= v_{\frac{k}{2}\frac{k}{2}o} \\ h_{\frac{k+n}{2}\frac{k-n}{2}p} &= v_{\frac{k+n}{2}\frac{k-n}{2}p} \end{aligned}\right. \tag{6.8}$$

in order to prevent $\sigma_{\frac{k}{2}\frac{k}{2}o}$ and $\sigma_{\frac{k+n}{2}\frac{k-n}{2}p}$ from becoming very large, which leads to

$$\left[\begin{aligned} \sigma_{\frac{k}{2}\frac{k}{2}o} &= o \\ \sigma_{\frac{k+n}{2}\frac{k-n}{2}p} &= o \end{aligned}\right. \tag{6.7b}$$

(In fact, only $\sigma_{\frac{k}{2}\frac{k}{2}o}$ = const. follows from (6.5) or (6.6). Putting the

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* Here the procedure differs from Moser's and Hagedorn's, who assume the perturbation Hamiltonian to contain only 3rd and higher order terms. Then the σ-coefficients appearing on the r.h.s. of equ. (6.5) and (6.6) are always of lower order than on the l.h.s. and the equations can be solved rigorously by recursion up to any degree (save for convergence limitations). Admitting also perturbations of degre < 3, the equations can only be solved by iterative approximations. This is no serious drawback from the point of view of practical usefulness, but, however a considerable advantage for the treatment of the linear perturbations.

const. equal to zero is again the most convenient choice).

The coefficients of the new Hamiltonian as given by (6.8) in first approximation correspond exactly to the zero and low frequency terms of the original Hamiltonian as had been assumed in section 5.

Reintroducing (6.7) and (6.8) into the r.h.s. of (6.6) we obtain a second approximation for the $\sigma_{\ell mq}$ and $h_{\ell mq}$ which will be correct up to second order terms in the coefficients $v_{\ell mq}$, as $\sigma_{\ell mq}$ and $h_{\ell mq}$ are linear in $v_{\ell mq}$ *in the first approximation. We need only the coefficients* $h_{\frac{k+n}{2}\,\frac{k-n}{2}\,p}$ in second approximation, which we find from (6.6) and the requirements on $\bar{H}$:

$$h_{\frac{k+n}{2}\,\frac{k-n}{2}\,p} = v_{\frac{k+n}{2}\,\frac{k-n}{2}\,p} - \sum_{q} \frac{i(\ell'-m')(\ell''+m'')}{2} \left(h^{(1)}_{\ell'm'q}\sigma^{(1)}_{\ell''m''(p-q)} - \sigma^{(1)}_{\ell'm'q}v_{\ell''m''(p-q)}\right)$$

where $h^{(1)}$, $\sigma^{(1)}$ indicates first approximation values. The n and p occuring in the suffixes satisfy $nQ - p \approx 0$.

Now $h^{(1)}_{\ell'm'q} \neq 0$ only if $(\ell' - m')\,Q \approx q$; then simultaneously

$$(\ell'' - m'')Q = \left[(\ell - m) - (\ell' - m')\right]\,Q = p - q$$

and therefore

$$\sigma_{\ell''m''(p-q)} = 0$$

Thus there is no contribution of the first part of the sum and we have

$$h_{\frac{k+n}{2}\,\frac{k-n}{2}\,p} = v_{\frac{k+n}{2}\,\frac{k-n}{2}\,p} - \frac{1}{2}\sum_{(\ell'-m')Q-q\neq 0} (\ell'-m')(\ell''+m'')\,\frac{v_{\ell'm'q}\,v_{\ell''m''(p-q)}}{(\ell'-m')Q-q}\,. \qquad (6.9)$$

The second order corrections to the transformed Hamiltonian are usually negligible in the application of this theory to an almost linear synchrotron. Furthermore, if they become important, the rapidly oscillating part of the motion becomes appreciable at the same time, as will be seen later in section 7. They may, however, have to be taken into account if some coefficient of the Hamiltonian vanishes in the first approximation or is very small. A few of the coefficients which will be of interest later on for this reason are, therefore, given here explicitly

$$\left[\begin{aligned} h_{110} &= v_{110} - 4 \sum_{q \neq 2Q} \frac{v_{20q}\, v_{02(-q)}}{2Q - q} \\ h_{20p} &= v_{20p} - 2 \sum_{q \neq 2Q} \frac{v_{20q}\, v_{11(p-q)}}{2Q - q} \\ h_{220} &= v_{220} - 3 \sum_{q \neq Q,2Q,3Q} \left\{ 3\, \frac{v_{30q}\, v_{03(-q)}}{3Q - q} + \frac{v_{21q}\, v_{12(-q)}}{Q - q} \right. \\ &\qquad \left. + 2\, \frac{v_{20q}\, \overset{*}{v}_{31q} + v_{31q}\, \overset{*}{v}_{20q}}{2Q - q} \right\} \end{aligned}\right. \tag{6.10}$$

7. The rapidly varying parts of phase and amplitude

For the power expansions, used above in deriving σ to be valid, the rapidly varying parts $\tilde{\varphi}$ and $\tilde{a}$ of phase and amplitude must be small in the following sense

$$\tilde{\varphi} \ll 1, \qquad \frac{\tilde{a}}{A} \ll 1. \tag{7.1}$$

$\tilde{\varphi}$ and $\tilde{a}$ follow from (6.1) :

$$\tilde{\varphi} = \varphi - \phi = - \frac{\partial\sigma}{\partial A}$$

$$\tilde{a} = a - A = \frac{\partial\sigma}{\partial\varphi}$$

Introducing σ from (6.4) and (6.7), we obtain in first approximation

$$\left[\begin{aligned} \tilde{\varphi} &= \sum_k \sum_{\ell+m=k} \sum_{q \neq (\ell-m)Q} \frac{k}{2} A^{\frac{k}{2}-1} \frac{v_{\ell mq}}{i[(\ell-m)Q-q]} e^{i[(\ell-m)(Q\vartheta + \phi) - q\vartheta]} \\ \tilde{a} &= \sum_k \sum_{\ell+m=k} \sum_{q \neq (\ell-m)Q} (\ell-m)\, A^{\frac{k}{2}} \frac{v_{\ell mq}}{(\ell-m)Q-q} e^{i[(\ell-m)(Q\vartheta + \phi) - q\vartheta]} \end{aligned}\right. \tag{7.2}$$

where the "low frequency" terms characterized by $(\ell-m)Q - q \approx 0$ have to be excluded from the sum, as $\sigma_{\ell mq} = 0$ for these combinations of ℓ,m,q.

(7.2) may be used to check the inequalities (7.1) in each individual case.

If the modulation of the Floquet factor is small, so that the free oscillations are smooth, the order of magnitude of one term of the sum making up $\tilde{\varphi}$ is

$$\binom{k}{m} \frac{w_o^k \; V_{kq}}{(\ell-m)Q-q} \frac{k}{2} A^{\frac{k}{2}-1} \approx \binom{k}{m} \frac{V_{kq}}{(\ell-m)Q-q} \frac{k}{2} \frac{A^{\frac{k-2}{2}}}{(2Q)^{\frac{k}{2}}} =$$

$$\binom{k}{m} \frac{Q}{(\ell-m)Q-q} \frac{1}{2^k} k \frac{R^{k-1} \; V_{kq}}{Q^2 R} = \binom{k}{m} \frac{1}{2^k} \frac{Q}{(\ell-m)Q-q} \frac{\frac{dR^k}{dR} V_{kq}}{Q^2 R}$$

where $R \approx \left(\frac{2A}{Q}\right)^{\frac{1}{2}}$ is the slowly varying part of the actual amplitude. A similar estimate is obtained for $\frac{\tilde{a}}{A}$.

Thus the term in question is proportional to the ratio

$$\frac{\frac{dR^k}{dR} V_{kq}}{Q^2 R}$$

of the q-th harmonic of the k-th power perturbing force to the effective unperturbed restoring force, taken for an elongation equal to the amplitude. The proportionality factor may be large if $|(\ell-m)Q-q|$ is small, that is, if the term is an almost slowly varying one. The ratio of perturbation to unperturbed force must, therefore, remain sufficiently small for the whole approximation to be useful. Numerical examples will be found in section 10.

8. Treatment of perturbations causing distortions of the "closed orbit"

The method described in the preceding sections encounters a difficulty if the perturbation Hamiltonian contains a term linear in x (which it will in the presence of guiding field errors or alignment errors in a synchrotron). The power $a^{\frac{1}{2}}$ will then appear in the transformed Hamiltonian and from (7.2) it follows that the rapidly oscillating parts $\tilde{\varphi}$ and $\frac{\tilde{a}}{A}$ become very large for $A \to 0$, thus invalidating the conditions for the approximation. The reason for this difficulty can easily be seen by considering the effect of a constant force displacing the equilibrium position. A constant or nearly constant displacement, however small it may be, cannot be represented by a free oscillation with modulated phase and amplitude the rapidly varying part of which is small compared to the amplitude itself.

The difficulty is removed by measuring the displacement from the distorted "closed orbit" which is that particular solution of the equations of motion (3.1) that is periodic with period 2π (or the fundamental period of the perturbation and $n(\vartheta)$). The determination of the closed orbit then remains a separate problem which has to be solved by a different method.

Assume $x = c(\vartheta)$ to be the perturbed closed orbit, then the displacement y with respect to the closed orbit is defined by

$$x = c(\vartheta) + y. \tag{8.1}$$

Substituting this in the Hamiltonian (3.1), and defining a conjugate momentum y' by

$$x' = c' + y', \quad * \tag{8.2}$$

the equations (3.1) become

$$\begin{aligned} \frac{dx}{d\vartheta} &= \frac{dc}{d\vartheta} + \frac{dy}{d\vartheta} = \frac{\partial}{\partial y'} H(c+y,\ c'+y') \\ \frac{dy'}{d\vartheta} &= \frac{dc'}{d\vartheta} + \frac{dy'}{d\vartheta} = -\frac{\partial}{\partial y} H(c+y,\ c'+y'). \end{aligned} \tag{8.3}$$

We now divide $H(c+y,\ c'+y')$ into two parts, the first comprising the terms linear in y and y', the second comprising all of the rest :

$$\left[\begin{aligned} H &= \left(\frac{\partial}{\partial c} H(c,\ c')\right) y + \left(\frac{\partial}{\partial c'} H(c,\ c')\right) y' + \mathcal{H} \\ \mathcal{H} &= H(c+y,\ c'+y') - \left[\frac{\partial H(c,c')}{\partial c}\, y + \frac{\partial H}{\partial c'}\, y'\right]. \end{aligned}\right. \tag{8.4}$$

As the closed orbit satisfies the original equations (3.1) :

---

* Remember that x', y', c' do not exactly equal the derivatives $\frac{dx}{d\vartheta}$, $\frac{dy}{d\vartheta}$, $\frac{dc}{d\vartheta}$, except for the unperturbed Hamiltonian.

$$\frac{dc}{d\vartheta} = \frac{\partial H(c,c')}{\partial c'}$$

$$\frac{dc'}{d\vartheta} = - \frac{\partial H(c,c')}{\partial c} \quad , \tag{8.5}$$

(8.3) reduces to Hamiltonian equations for y :

$$\frac{dy}{d\vartheta} = \frac{\partial \mathcal{H}}{\partial y'}$$

$$\frac{dy'}{d\vartheta} = -\frac{\partial \mathcal{H}}{\partial y} \tag{8.6}$$

where $\mathcal{H}$ as given by (8.4) and, expanded in polynomials in y and y', no longer contains any linear terms in y and y'.

If the perturbation part $H^{(1)}$ of the original Hamiltonian does not depend on x' (i.e. if the assumptions made in (4.2) are valid) the same is true of the perturbation part $\mathcal{H}^{(1)}$ of $\mathcal{H}$, and $\mathcal{H}^{(1)}$ is then a polynomial in y only.

Although the method of variation of amplitude and phase has proved disadvantageous for solving the equations (8.5), we still make use of the φ,a-representation (3.2) to define "smooth motion" coordinates $\bar{x}$, $\bar{x}'$ by

$$\bar{x} = \sqrt{\frac{a}{2Q}} \left(e^{i(Q\vartheta+\varphi)} + e^{-i(Q\vartheta+\varphi)} \right) = \sqrt{\frac{2a}{Q}} \cos (Q\vartheta+\varphi)$$

$$\bar{x}' = iQ\sqrt{\frac{a}{2Q}} \left(e^{i(Q\vartheta+\varphi)} - e^{-i(Q\vartheta+\varphi)} \right) = - Q \sqrt{\frac{2a}{Q}} \sin (Q\vartheta+\varphi), \tag{8.7}$$

i.e. as the coordinates of an oscillator of frequency Q with amplitude $\sqrt{\frac{2a}{Q}}$ and phase φ. In this way the A.G. structure wriggle is separated off. In order to express the Hamiltonian now in terms of $\bar{x}$, $\bar{x}'$ we observe that the transformation (8.7) can be generated by a function

$$S(\varphi,\bar{x}', \vartheta) = -\frac{\bar{x}'^2}{2Q} \operatorname{ctg} (Q\vartheta+\varphi)$$

as

$$\bar{x} = \frac{\partial S}{\partial \bar{x}'} = - \frac{\bar{x}'}{Q} \operatorname{ctg} (Q\vartheta+\varphi)$$

$$a = \frac{\partial S}{\partial \varphi} = \frac{\bar{x}'^2}{2Q} \frac{1}{\sin^2(Q\vartheta+\varphi)}$$

which is equivalent to (8.7). The new Hamiltonian is, therefore,

$$H^{(1)} + \frac{\partial S}{\partial \vartheta} = H^{(1)} + \frac{\overline{x'}^2}{2\sin^2(Q\vartheta+\varphi)} \cdot$$

φ, a have to be replaced by $\bar{x}$, $\bar{x}'$ by means of (8.7). The second term then becomes

$$\frac{\overline{x'}^2}{2\sin^2(Q\vartheta+\varphi)} = Qa = \frac{Q^2\bar{x}^2 + \overline{x'}^2}{2}.$$

If we start from $H^{(1)}(x,x',\vartheta)$ in the original variables, we may transform immediately from x,x' to $\bar{x},\bar{x}'$ by combining (3.2) and (8.7)

$$x = \sqrt{\frac{Q}{2}}\left[(w_1 + \overset{*}{w}_1)\,\bar{x} - \frac{i}{Q}\,(w_1 - \overset{*}{w}_1)\,\overline{x'}\right]$$

$$x' = \sqrt{\frac{Q}{2}}\left[(w_2 + \overset{*}{w}_2)\,\bar{x} - \frac{i}{Q}\,(w_2 - \overset{*}{w}_2)\,\overline{x'}\right] \tag{8.8}$$

After this substitution we find as equations of motion for $\bar{x}$, $\bar{x}'$

$$\frac{d\bar{x}}{d\vartheta} = \bar{x}' + \frac{\partial H^{(1)}}{\partial \overline{x'}}$$

$$\frac{d\overline{x'}}{d\vartheta} = -Q^2\bar{x} - \frac{\partial H^{(1)}}{\partial \bar{x}}$$

or

$$\frac{d^2\bar{x}}{d\vartheta^2} + Q^2\bar{x} = -\frac{\partial H^{(1)}}{\partial \bar{x}} + \frac{d}{d\vartheta}\frac{\partial H^{(1)}}{\partial \overline{x'}} \, . \tag{8.9}$$

This equation is rigorously equivalent to the original equation of motion (3.1) resp. (8.5). By introducing the smooth motion coordinates it has become that of a simple harmonic oscillator exposed to linear and non-linear perturbations given by the terms on the right hand side. For calculation of the closed orbit (8.9) is better suited than (8.5). Examples of how to use (8.9) to obtain good approximations to closed orbits in A.G. synchrotrons are given in more detail in Appendix V.

9. General discussion of the motion

If Q is rational or almost rational,

$$Q \approx \frac{p}{n} \, , \tag{9.1}$$

where p,n the smallest integers possible, the <u>slowly varying Hamiltonian</u> according to sections 5 and 6 is explicitly :

$$\bar{H} = \sum_{k \geq 2} h_{\frac{k}{2}\frac{k}{2}0} A^{\frac{k}{2}}$$

$$+ \sum_{k \geq n} A^{\frac{k}{2}} \left\{ h_{\frac{k+n}{2}\frac{k-n}{2}p}\, e^{in[(Q - \frac{p}{n})\vartheta + \phi]} + h_{\frac{k-n}{2}\frac{k+n}{2}(-p)}\, e^{-in[(Q - \frac{p}{n})\vartheta + \phi]} \right\}$$

$$+ \sum_{k \geq 2n} A^{\frac{k}{2}} \left\{ h_{\frac{k+2n}{2}\frac{k-2n}{2}2p}\, e^{i2n[(Q-\frac{p}{n})\vartheta+\phi]} + h_{\frac{k-2n}{2}\frac{k+2n}{2}(-2p)}\, e^{-i2n[(Q-\frac{p}{n})\vartheta+\phi]} \right\}$$

$$+ \dots \qquad (9.2)$$

No terms of degree $k = 1$ have been included. If they occur in the original Hamiltonian the present Hamiltonian is supposed to be derived from $\mathcal{H}$ of *section 8, valid for the oscillations about the distorted closed orbit.* The coefficients $h_{\ell mq}$ are given by (6.8) in first, by (6.9) in second approximation. As the suffices of h are integers ≥ 0, n and k are even or odd simultaneously in the appearing coefficients, and the summations start with $k \geq n$, $k \geq 2n$ ... etc.

The equations of motion following from (9.2) are

$$\frac{d\phi}{d\vartheta} = \frac{\partial \bar{H}}{\partial A} = \sum_{k \geq 2} \frac{k}{2} h_{\frac{k}{2}\frac{k}{2}o} A^{\frac{k}{2}-1}$$

$$+ \sum_{k \geq n} \frac{k}{2} h_{\frac{k+n}{2}\frac{k-n}{2}p} A^{\frac{k}{2}-1} e^{in[(Q-\frac{p}{n})\vartheta + \phi]} + \text{conj. complex} \qquad (9.3)$$

$$+ \sum_{k \geq 2n} \frac{k}{2} h_{\frac{k+2n}{2}\frac{k-2n}{2}2p} A^{\frac{k}{2}-1} e^{i2n[(Q-\frac{p}{n})\vartheta + \phi]} + \text{conj. complex}$$

$$+ \dots$$

$$\frac{dA}{d\vartheta} = -\frac{\partial \bar{H}}{\partial \phi} = -\,in \sum_{k \geq n} h_{\frac{k+n}{2}\frac{k-n}{2}p} A^{\frac{k}{2}} e^{in[(Q-\frac{p}{n})\vartheta + \phi]} + \text{conj. complex}$$

$$-\, i2n \sum_{k \geq 2n} h_{\frac{k+2n}{2}\frac{k-2n}{2}2p} A^{\frac{k}{2}} e^{i2n[(Q-\frac{p}{n})\vartheta + \phi]} + \text{conj. complex}$$

$$- \dots$$

If $\bar{H}$ comprises only zero frequency terms $n = 0$, $p = 0$, either because the relevant coefficients $h_{\frac{k+n}{2}\,\frac{k-n}{2}\,p}$ etc in (9.2) are zero (i.e. there are no ϑ-dependent perturbations), or because Q is not nearly rational, the equations of motion become simpler

$$\frac{d\phi}{d\vartheta} = \sum_{k \geqq 2} \frac{k}{2}\, h_{\frac{k}{2}\,\frac{k}{2}\,o}\, A^{\frac{k}{2}-1} = h_{110} + 2\, h_{220}\, A + \dots \tag{9.4}$$

$$\frac{dA}{d\vartheta} = 0.$$

They show that the amplitude is constant (in the average, i.e. apart from the rapidly oscillating part). This is the case of a <u>free oscillation without excitation</u>. Obviously $\frac{d\phi}{d\vartheta}$ is a correction to the frequency due to the perturbation, the effective frequency being $Q + \frac{d\phi}{d\vartheta}$. The linear perturbations (of degree $k = 2$ in the Hamiltonian) produce a frequency shift h_{110} independent of amplitude. The terms $2h_{220}\, A + \dots$ following give the shift of frequency dependent on amplitude due to non-linearities. The contribution $2h_{220}\, A$ is proportional to the square of the amplitude; a look at (6.10) shows that h_{220} is, in first approximation, caused by 4th order terms in the Hamiltonian (cubic non-linearities in the forces), and that 3rd order terms (quadratic non-linearities) contribute in second approximation.

Returning to equations (9.3) a change in amplitude is caused by the terms appearing on account of a rational value of $Q \approx \frac{p}{n}$. These terms are therefore, responsible for the <u>excitation of a "subresonance"</u> ; n may be called the order of the subresonance ($n = 1$ corresponding to an ordinary resonance).

Suppose that the phase ϕ, at some instant, is such that the change of amplitude is a growth. How far the growth will go on depends on the change of phase due to the first of the equations (9.3). In general, the phase will, after some time, have changed enough to cause the amplitude to decrease again. The motion then is a beating oscillation. Under certain conditions, there may be unlimited build-up, if e.g. (i) the coefficients h_{220}, $h_{330} \dots$ are all zero so that there is no shift of frequency with amplitude, and (ii) only one of the exciting coefficients $h_{\frac{k+n}{2}\,\frac{k-n}{2}\,p} = |h|\, e^{in\beta}$ is different from zero (we do not consider more general possibilities here). The equations of motion then take the more particular form

$$\frac{d\phi}{d\vartheta} = h_{110} + 2|h| \frac{k}{2} A^{\frac{k}{2}-1} \cos n[(Q - \frac{p}{n}) \vartheta + \phi + \beta]$$

$$\frac{dA}{d\vartheta} = 2|h| n A^{\frac{k}{2}} \sin n [(Q - \frac{p}{n}) \vartheta + \phi + \beta] \tag{9.5}$$

The maximum rate of growth of A is obtained for $\sin n[\ldots] = 1$. This condition is maintained automatically by virtue of the first of these equations,

$$\frac{d\phi}{d\vartheta} = h_{110} = \text{const},$$

if

$$Q + h_{110} - \frac{p}{n} = 0,$$

i.e. if the effective frequency satisfies the condition for subresonance.

It is obvious from this, that any terms shifting the frequency with amplitude tend to limit the build-up of amplitude, transforming it into a mere beating of the oscillation.

For the slowly varying part of the motion an <u>invariant</u> relation between amplitude and phase can be established which is very useful for studying the range of amplitudes covered by the oscillation in the course of its beating. This invariant had been derived and used by Beth [1910], Krylov and Bogoliubov [1937], Judd [1950], Moser [1955] and Sturrock [1955]. It follows most easily by calculating the rate of change of the Hamiltonian $\bar{H}$ (see (9.2)) which depends on ϑ explicitly :

$$\frac{d\bar{H}}{d\vartheta} = \frac{\partial\bar{H}}{\partial\vartheta} = in\,(Q - \frac{p}{n})\, e^{in[(Q - \frac{p}{n})\vartheta + \phi]} \sum_{k \geq n} h_{\frac{k+n}{2}\,\frac{k-n}{2}\,p} A^{\frac{k}{2}} + \text{conj. compl.}$$

$$+ i2n(Q - \frac{p}{n})\, e^{i2n[(Q - \frac{p}{n})\vartheta + \phi]} \sum_{k \geq 2n} h_{\frac{k+2n}{2}\,\frac{k-2n}{2}\,2p} A^{\frac{k}{2}} + \text{conj. compl.}$$

$$+ \ldots$$

$$= (Q - \frac{p}{n}) \frac{\partial\bar{H}}{\partial\phi} = -(Q - \frac{p}{n}) \frac{dA}{d\vartheta} ,$$

where (9.3) has been used. Thus

$$C = (Q - \frac{p}{n}) A + \bar{H} = (Q + h_{110} - \frac{p}{n}) A + h_{220} A^2 + \dots$$

$$+ \left\{ \sum_{k \geqq n} h_{\frac{k+n}{2} \frac{k-n}{2} p} A^{\frac{k}{2}} e^{in[(Q - \frac{p}{n})\vartheta + \phi]} + \text{conj. complex} \right\}$$

$$+ \left\{ \sum_{k \geqq 2n} h_{\frac{k+2n}{2} \frac{k-2n}{2} 2p} A^{\frac{k}{2}} e^{i2n[(Q - \frac{p}{n})\vartheta + \phi]} + \text{conj. complex} \right\}$$

$$+ \dots \qquad (9.6)$$

is a constant of the motion. Its value is, of course, determined by the initial amplitude and phase. It will be employed in the following section to obtain phase space diagrams and beating ranges of the motion for the subresonances of order 2 to 5.

To conclude this section it may be noted that the equations of motion (9.3) can be integrated completely by means of the invariant (9.6), in principle at least : Solving (9.6) for

$$e^{in[(Q - \frac{p}{n}) \vartheta + \phi]}$$

in terms of A and C, and substituting in the Hamiltonian equation (9.3) for A we have

$$\frac{dA}{d\vartheta} = - \frac{\partial \bar{H}}{\partial \phi} = \text{function of } (A, C).$$

From this $\vartheta = \vartheta(A, C)$ may be found by straightforward integration.

10. Detailed discussion of the subresonances of order n=2 to n=5

The invariant relation (9.6) determines A as a function of $(Q - \frac{p}{n})\vartheta + \phi$, that is the path in the two dimensional phase space of the system. It should be noted that the phase path obtained in this way does not represent the full motion of the original system but only what is left after removal of the Floquet modulation, the oscillatory motion proper, and the rapid variations of amplitude and phase. Thus only the evolution of the mean amplitude and phase is retained.

To simplify detailed discussion we assume <u>one exciting term only,</u> with coefficients

$$h_{no\pm p} = |h_{nop}| e^{\pm in\beta}$$

(this is the lowest degree term sufficient to excite the n-th order subresonance), and one fixed non-linearity h_{220}.

Then

$$C = (Q + h_{110} - \frac{p}{n}) A + h_{220} A^2 + 2|h_{nop}| A^{\frac{n}{2}} \cos n[(Q - \frac{p}{n})\vartheta + \phi + \beta] \tag{10.1}$$

The evaluation of C in its general form (9.6) would be considerably more troublesome. The restriction made does not affect qualitative features and orders of magnitude.

By <u>measuring the amplitude in units of some reference amplitude</u> given by A_0,* we may introduce the dimensionless variable

$$\alpha = \frac{A}{A_0} \tag{10.2}$$

and write

$$C' = \Delta Q\alpha + \frac{\Delta Q_s}{2} \alpha^2 + \frac{2}{n} \Delta Q_e \alpha^{\frac{n}{2}} \cos n \psi \tag{10.3}$$

where

$$\psi = (Q - \frac{p}{n})\vartheta + \phi + \beta, \tag{10.4}$$

$$\Delta Q = Q + h_{110} - \frac{p}{n} \tag{10.5}$$

the "detuning" , equal to the difference of the small oscillation frequency $Q + h_{110}$ from the nominal subresonance frequency $\frac{p}{n}$,

$$\Delta Q_s = 2h_{220} A_0 \tag{10.6}$$

the non-linear frequency shift at the reference amplitude, and

---

* Remember that the actual amplitude is

$$R = 2A^{\frac{1}{2}} w_0 \approx \left(\frac{2A}{Q} \right)^{\frac{1}{2}}$$

according to section 3.

$$\Delta Q_e = n|h_{nop}|A_o^{\frac{n}{2}-1} \tag{10.7}$$

the "excitation width". It will be seen later how the strength of excitation is related to the width of the frequency interval where oscillations are unstable (in non-stabilized systems).

To reduce the numbers of parameters to its minimum we divide (10.3) by $\frac{2}{n}\Delta Q_e$ and write

$$C = \xi\alpha + \kappa\alpha^2 + \alpha^{\frac{n}{2}} \cos n\psi \tag{10.8}$$

where

$$\xi = \frac{\Delta Q}{\frac{2\Delta Q_e}{n}} = \frac{\Delta Q}{2|h_{nop}|A_o^{\frac{n}{2}-1}} \tag{10.9}$$

$$\kappa = \frac{\Delta Q_s}{\frac{2\Delta Q_e}{n}} = \frac{h_{220}}{2|h_{nop}|A_o^{\frac{n}{2}-2}} \tag{10.10}$$

measure detuning and frequency shifting non-linearity in terms of the excitation width, and are the natural parameters of the problem.

Figs. 1 to 4 show phase diagrams $\sqrt{\alpha} = \sqrt{\alpha(\psi)}$ as following from the invariant (10.8) in polar coordinates for the vicinity of subresonances of order $n = 2, 3, 4, 5$. For each subresonance sets of curves without frequency shifting non-linearity ($\kappa = 0$), and with a frequency shifting non-linearity ($\kappa = 2$) are given. Such phase plots had been used frequently before in discussing linear and non-linear oscillations; therefore, only a few points shall be noted here : First the phase paths are either closed (i.e. amplitude and phase change periodically*), or they reach infinity representing respectively stable or unstable motions. The latter is the case for $\kappa = 0$ and $\xi = 0$ (or very nearly zero). Going from $\kappa = 0$ to $\kappa \neq 0$, the unstable paths are made to close at finite amplitudes and to become stable. For subresonances orders $n \geqq 4$ this is true for small enough amplitudes only as the diagrams show. Large amplitudes remain unstable for this orders. It should be remembered that a practical system will in general have more non-linear terms than have been assumed for simplification in (10.1). The topology of the phase paths at larger amplitudes will then be affected by the non-linear

* This may be true only in the approximation used. In full rigour, the phase curves might show ergodic diffusion. See relevant remarks of Moser [1955].

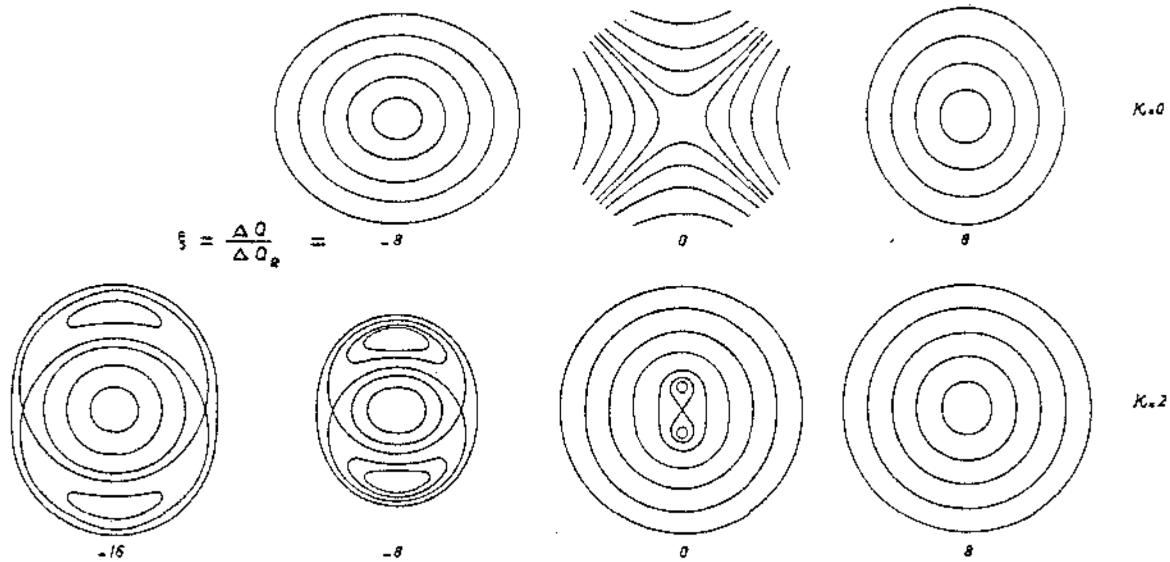

Fig. 1 Phase plane paths in the vicinity of subresonances of order n = 2.

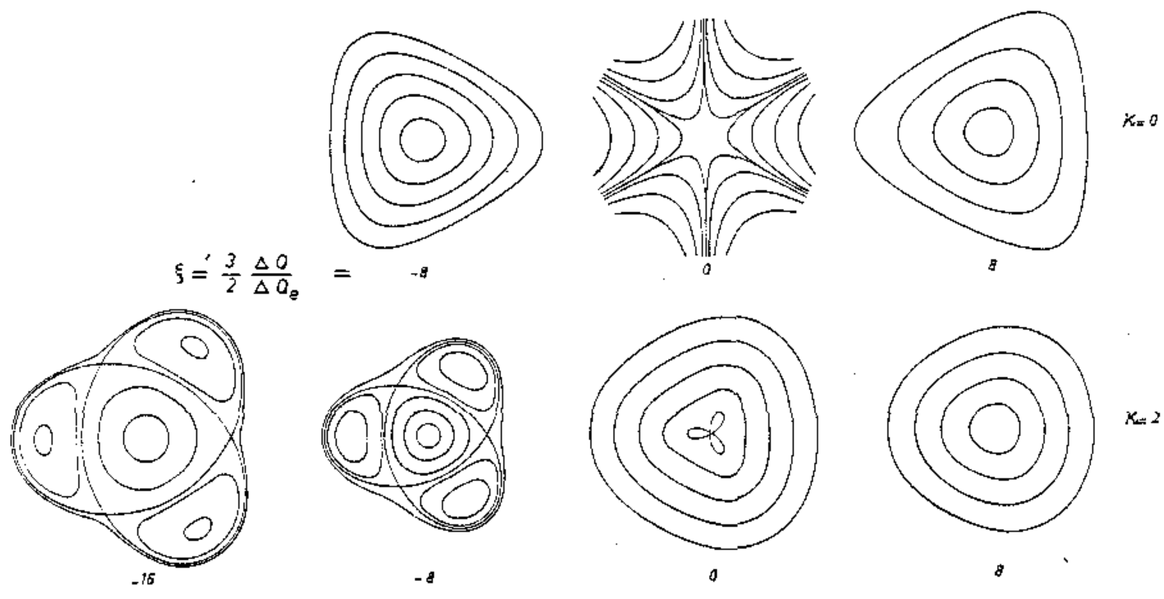

Fig. 2 Phase plane paths in the vicinity of subresonances of order n = 3

facing p. 26

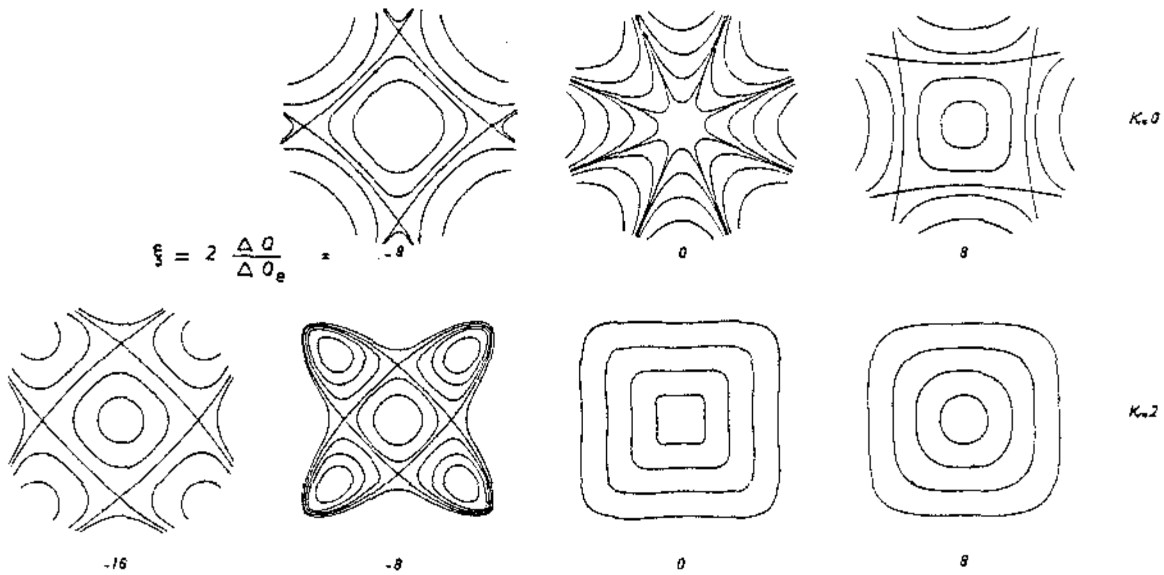

Fig. 3 Phase plane paths in the vicinity of subresonances of order n = 4.

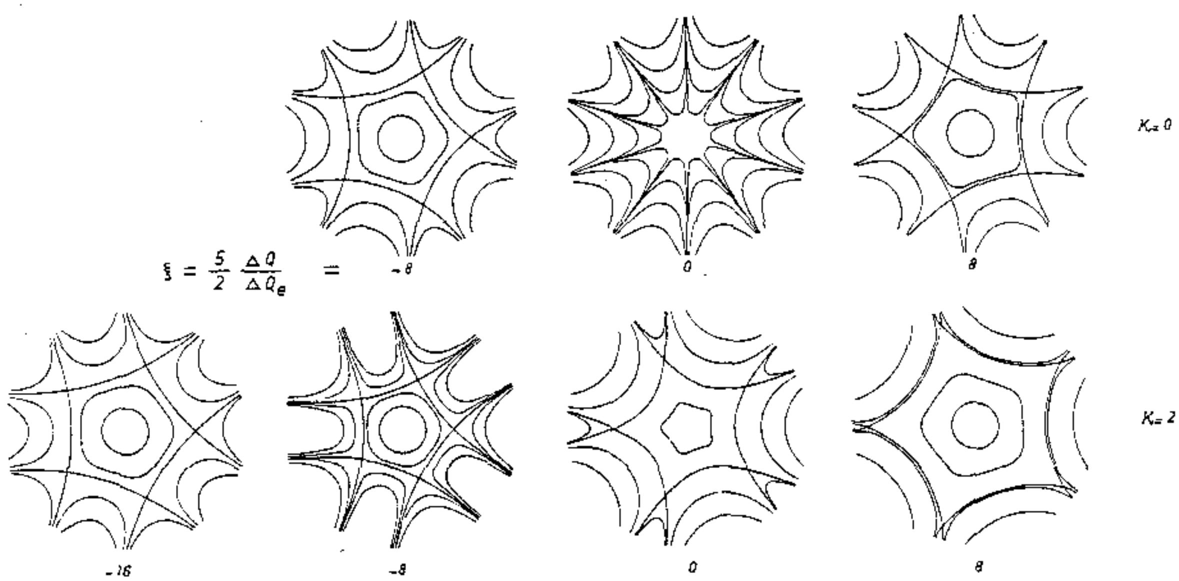

Fig. 4 Phase plane paths in the vicinity of subresonances of order n = 5.

facing p. 27

terms of higher degree.

Further, note the appearance of "islands" outside the zero point in certain stabilized cases. The centers of the islands are "fixed points" which represent motions with fixed amplitude and phase. From

$$\frac{d\psi}{d\vartheta} = Q - \frac{p}{n} + \frac{d\phi}{d\vartheta} = 0 \qquad (10.11)$$

it follows that in a fixed point the perturbed frequency $Q + \frac{d\phi}{d\vartheta}$ is exactly rational $= \frac{p}{n}$. In other words, the corresponding orbits are "closed orbits" which close after n revolutions.

The quantity of practical interest with regard to the confinement of particles by the focusing field of a synchrotron is the range of amplitudes covered by the oscillation. As illustrated by the phase plane diagrams, it depends on the ratio κ of non-linear frequency shift to strength of excitation and on the value of the invariant C, that is on the particular phase curve. In function of the detuning ΔQ, the beating range is more conveniently investigated by the type of diagram given in fig. 5 for the subresonance of order $n = 3$. There, C according to (10.8) is plotted as function of α for fixed phases; in fact only the limiting curves with $\cos n\psi = \pm 1$ are drawn. Several such pairs, corresponding to different values of the detuning parameter $\xi = \frac{n}{2} \frac{\Delta Q}{\Delta Q_e}$ are shown. The amplitude range is that part of the line C = const. which is cut off by the relevant pair of limiting curves.

We now ask for the maximum value α_{max} of α that can be reached from the initial value $\alpha_0 = 1$ by suitable choice of the initial phase (we can always make $\alpha_0 = 1$ by choosing the initial amplitude as reference amplitude in the definition (10.2) of α). In a case as designated by [1] and [3] in fig. 5, the initial phase leading to α_{max} is on one of the limiting curves. It may, however, be intermediate as in a case like [2] on fig.5. To avoid the tedious calculation of the exact α_{max} in the latter case we shall satisfy ourselves, in what follows, by finding the amplitude limits on the dashed lines which are upper and lower bounds to the actual maximum. So we have to do only with lines C = const., which at $\alpha = 1$ are intersected by one of the limiting curves $\cos n\psi = \pm 1$. On these lines $\sqrt{\alpha_{max}}$ is the ratio of maximum to minimum amplitude of the motion, sometimes called "beating factor".

If we start from $\cos n\psi = + 1$ at $\alpha = 1$, we denote the other

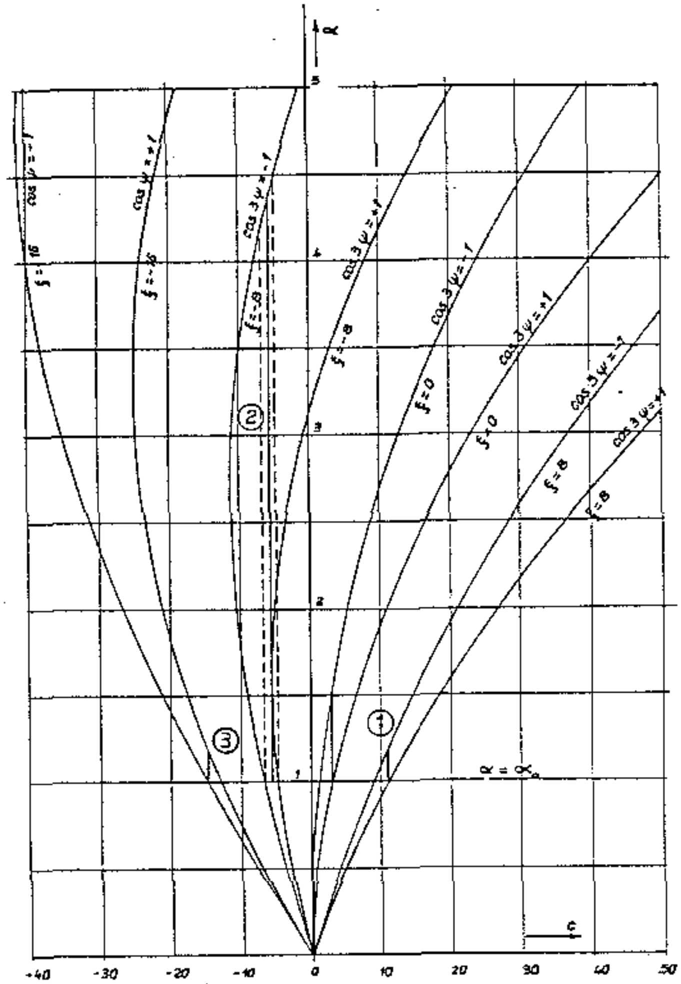

Fig. 5 Maximum beating ranges in a graphical representation of $C = \xi\alpha + \kappa\alpha^2 \pm \alpha^{3/2}$ (for 3rd order subresonance, $\kappa = 2$).

intersections of C = const. with the curves $\cos n\psi = \pm 1$ by α_{++}, α_{+-} respectively. Similarly, if we start from $\cos n\psi = -1$, the other intersections are α_{-+}, α_{--}. These values of α are determined by the equations

$$C = \xi\alpha_{++} + \kappa\alpha_{++}^2 + \alpha_{++}^{\frac{n}{2}} = \xi + \kappa + 1$$

$$C = \xi\alpha_{+-} + \kappa\alpha_{+-}^2 - \alpha_{+-}^{\frac{n}{2}} = \xi + \kappa + 1$$

$$C = \xi\alpha_{-+} + \kappa\alpha_{-+}^2 + \alpha_{-+}^{\frac{n}{2}} = \xi + \kappa - 1$$

$$C = \xi\alpha_{--} + \kappa\alpha_{--}^2 - \alpha_{--}^{\frac{n}{2}} = \xi + \kappa - 1,$$

which we solve for ξ to find how the α's depend on ξ :

$$\left.\begin{aligned}
\xi &= -\kappa\,(\alpha_{++} + 1) - \frac{\alpha_{++}^{\frac{n}{2}} - 1}{\alpha_{++} - 1} \\
\xi &= -\kappa\,(\alpha_{+-} + 1) + \frac{\alpha_{+-}^{\frac{n}{2}} + 1}{\alpha_{+-} - 1} \\
\xi &= -\kappa\,(\alpha_{-+} + 1) - \frac{\alpha_{-+}^{\frac{n}{2}} + 1}{\alpha_{-+} - 1} \\
\xi &= -\kappa\,(\alpha_{--} + 1) + \frac{\alpha_{--}^{\frac{n}{2}} - 1}{\alpha_{--} - 1}
\end{aligned}\right]
\qquad (10.12)$$

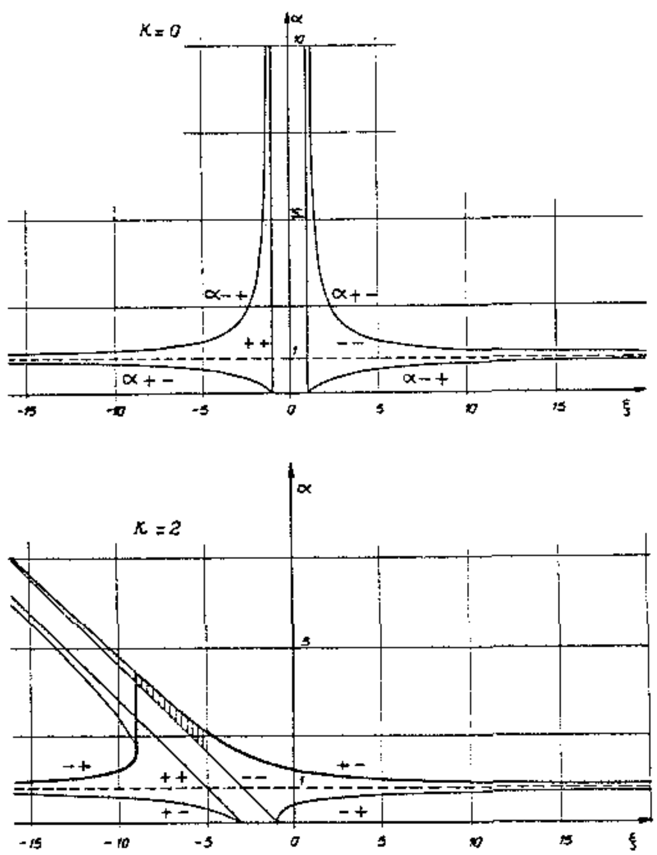

Fig. 6 Maximum beating ranges α as function of detuning, for subresonances of order $n = 2$ (top : $\kappa = 0$, no non-linear shift of frequency with amplitude; bottom : $\kappa = 2$, non-linear shift of frequency with amplitude).

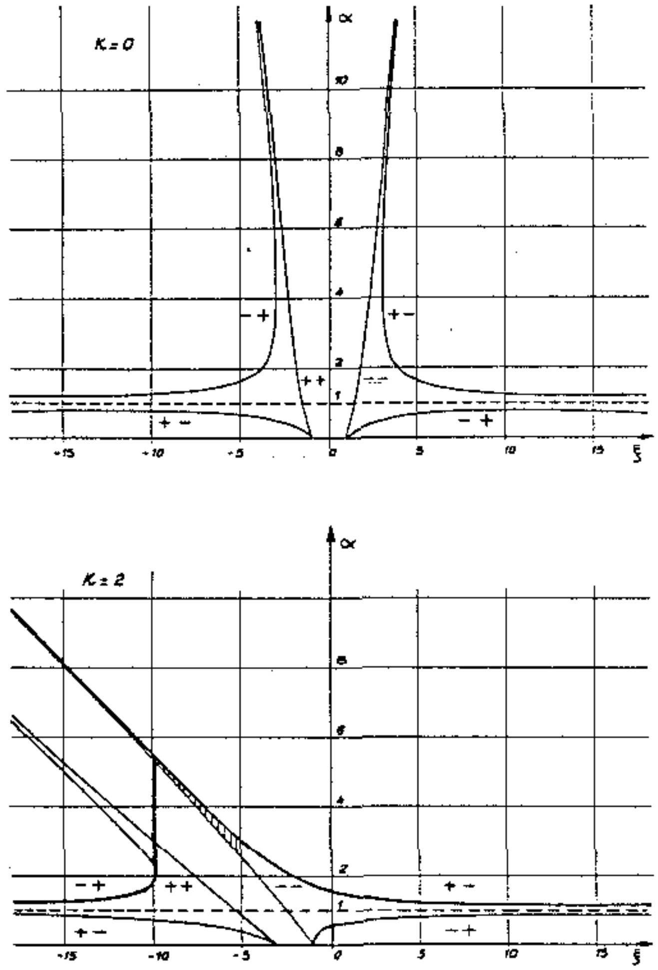

Fig. 7 Maximum beating ranges α as function of detuning, for subresonances of order $n = 3$ (top : $\kappa = 0$, no non-linear shift of frequency with amplitude; bottom : $\kappa = 2$, non-linear shift of frequency with amplitude).

These equations are plotted in a ξ, α plane, first for the linear subresonance $n = 2$, in fig. 6, for $\kappa = 0$ in the top part, and $\kappa = 2$ in the bottom part. The branches below $\alpha = 1$ belong to motions towards amplitudes smaller than the initial amplitude and are not of interest for our present question. For $\kappa = 0$, the upper branches go to infinity asymptotically as ξ approaches $+ 1$ from the right hand side or $- 1$ from the left hand side. Within the interval $- 1 < \xi < + 1$, there is no limitation of amplitude. This interval is known as "stopband" from linear theory. The width of the stopband is $\Delta\xi = 2$, or

$$\Delta Q_{stop} = 2\Delta Q_e$$

with regard to (10.9). This explains the term "excitation width" introduced for ΔQ_e in (10.7). For the $n = 2$ subresonance, ΔQ_e is independent of the initial amplitude according to (10.7).

For $\kappa \neq 0$, the curves are bent to one side by superposition of the straight line $-\kappa(\alpha + 1)$, see fig. 6, bottom part (p. 29). The bent-over leads to a limitation of amplitudes for all ξ, the stopband has disappeared. The heavy lines can be traced to represent the accessible maxima of α. In the shaded region, we have the conditions marked by [2] in fig. 5. The maximum is inside the shaded region, on its lower boundary at the left end, and on the upper boundary at the right end.

The same kind of curves are shown in figs. 7 and 8 for the $n = 3$ and $n = 5$ subresonances. There are again stopbands for $\kappa = 0$, whose width

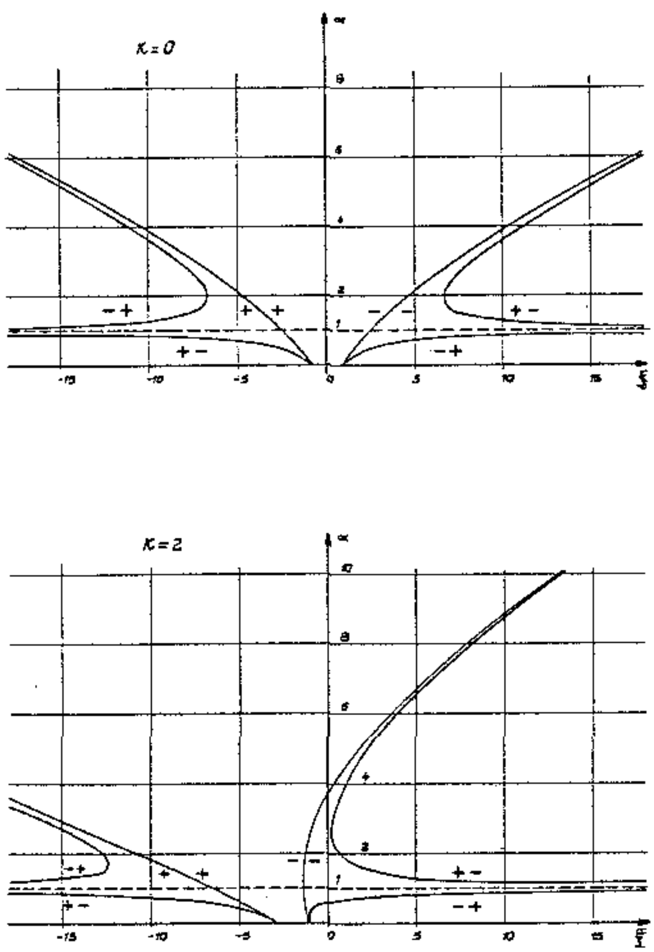

Fig. 8 Maximum beating ranges α as function of detuning, for subresonances of order $n = 5$ (top : $\kappa = 0$, no non-linear shift of frequency with amplitude; bottom : $\kappa = 2$, non-linear shift of frequency with amplitude).

is equal to the distance between the intersections of + + and - - curves with $\alpha = 1$, and is found from the figures or equations (10.12) to be

$$\Delta\xi = n$$

or, by (10.9) $\Delta Q_{stop} = 2\Delta Q_e$.
ΔQ_e is now, however, dependent on the initial amplitude for a given exciting perturbation, according to (10.7)

$$\Delta Q_e = n|h_{nop}| A_o^{\frac{n}{2}-1} \propto R_o^{n-2} .$$

By having $\kappa \neq 0$, the stopband is suppressed in the case of the $n = 3$ subresonance by bent-over of the curves, much like in the $n = 2$ case. For $n \geqq 4$, however, it depends on the magnitude of κ whether the bent-over is *sufficient to limit the amplitude range or not : inspection of (10.12) shows* that overcompensation of the second terms by the first term $- \kappa(\alpha + 1)$ is always possible for $n = 2$ and $n = 3$, and conditionally possible for $n \geqq 4$, the condition being $| \kappa | >$ some value depending on n. As by (10.10)

$$| \kappa | \propto \frac{1}{A_o^{\frac{n}{2}-2}} \propto \frac{1}{R_o^{n-4}}$$

$|\kappa|$ increases with increasing initial amplitude for $n < 4$, with decreasing initial amplitude for $n > 4$, and is independent of amplitude for $n = 4$. Thus with any $\kappa \neq 0$, subresonances $n > 4$ are always stable for sufficiently small amplitudes. For $n = 4$, stability requires $|\kappa| > 1$, independent of *amplitude. These properties seem to have been stated first by Moser* [1955] .

The examples of beating range vs frequency curves given in figs. 6 to 8 show the characteristic shape of non-linear resonance curves in the presence of a stabilizing non-linearity $\kappa \neq 0$. The maximum beating, occurring where the resonance curves show a sudden jump, depends on $|\kappa|$ as shown on fig. 9 (p. 33), for the subresonances n = 2, 3, 4, 5. Actually both the α_{+-} and α_{--} values at the ξ- value of the jump are plotted which include the real maximum as was pointed out by means of fig. 5. The difference between α_{+-} and α_{--} is of no practical significance.

The results of the theory regarding <u>practical stability</u> of betatron *oscillations near subresonances are now condensed in fig. 9. In particular*

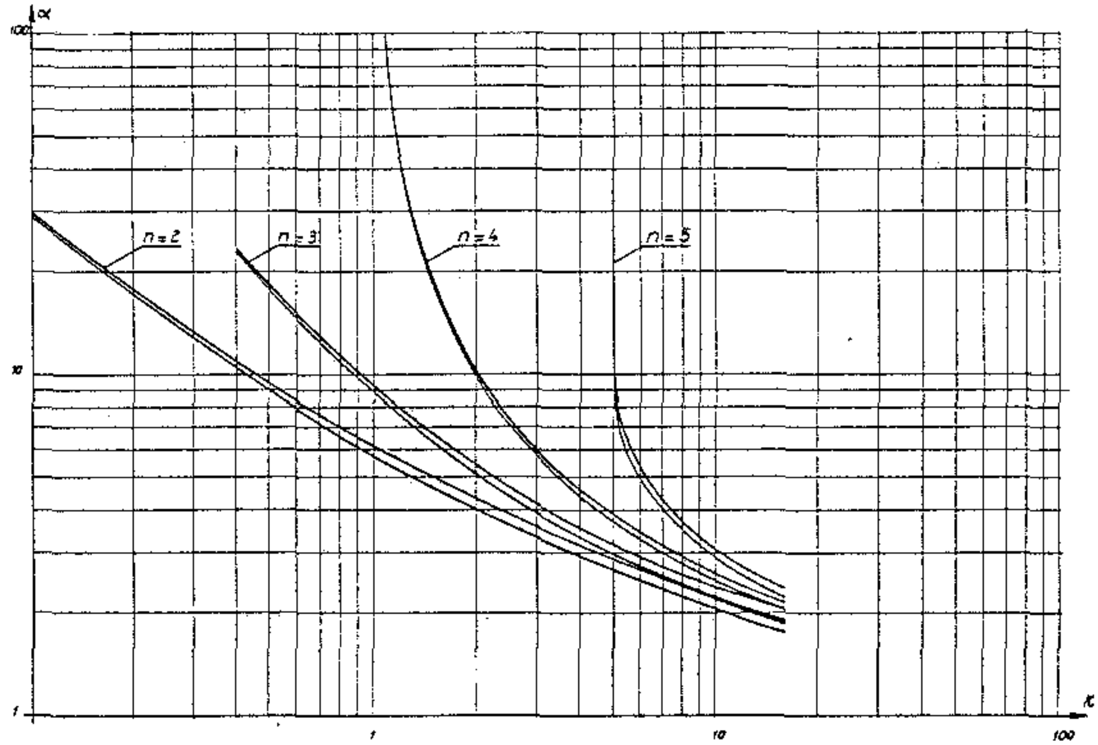

Fig. 9 α = (maximum of beating factor)² as function of stabilization parameter κ for subresonances of orders n = 2, 3, 4, 5.

one may conclude that a ratio of stabilizing non-linearity to excitation of $\kappa = 10 \ldots 20$ is required to keep $\alpha_{max} < 2$, that is the beating factor below $\sqrt{2}$.

It must be stressed at this point that these results apply to "static" conditions, where the parameters of the system do not vary with time. They may be seriously modified under "dynamic" conditions if the system is slightly varying; we shall return to this later.

11. Accuracy of results

For the method used in the foregoing sections to be a good approximation, the rapidly oscillating parts of phase and square-of-amplitude must be small, and it was pointed out in section 7 how to check this in particular cases by means of first order approximations for $\tilde{\varphi}$ and $\frac{\tilde{a}}{\bar{A}}$. A few examples may serve to show the restrictions imposed by this condition on the magnitude of the perturbations, for which the method is applicable.

First we try the method on a case for which the rigorous solution is known : the harmonic oscillator

$$\frac{dx}{d\vartheta^2} + (Q^2 + 2V)x = 0,$$

with a linear perturbation $2Vx$ independent of ϑ.

The solution is of course a harmonic oscillation with frequency $Q + \delta Q$, where

$$\delta Q = \sqrt{Q^2 + 2V} - Q = Q\left(\frac{V}{Q^2} - \frac{1}{2}\frac{V^2}{Q^4} + \ldots\right). \tag{11.1}$$

According to section 3, the variables φ, a are defined by

$$x = \sqrt{\frac{2a}{Q}}\cos(Q\vartheta + \varphi) = B\cos\left[(Q + \delta Q)\vartheta\right]$$

$$x' = -Q\sqrt{\frac{2a}{Q}}\sin(Q\vartheta + \varphi) = -B(Q + \delta Q)\sin\left[(Q + \delta Q)\vartheta\right]$$

where the amplitude B is arbitrary. Solving for a and φ yields

$$\frac{2a}{Q} = B^2\left\{1 + \frac{\delta Q}{Q} + \frac{1}{2}\left(\frac{\delta Q}{Q}\right)^2 - \left[\frac{\delta Q}{Q} + \frac{1}{2}\left(\frac{\delta Q}{Q}\right)^2\right]\cos\left[2(Q + \delta Q)\vartheta\right]\right\}$$

$$= B^2\left\{1 + \frac{V}{Q^2} - \frac{V}{Q^2}\cos\left[2(Q + \delta Q)\vartheta\right]\right\} \tag{11.2}$$

$$\operatorname{tg}(Q\vartheta + \varphi) = \left(1 + \frac{\delta Q}{Q}\right)\operatorname{tg}\left[(Q + \delta Q)\vartheta\right] \tag{11.3}$$

Writing $\varphi = \phi + \tilde{\varphi}$, with ϕ defined by $\frac{d\phi}{d\vartheta} = \delta Q$, we have from (11.3)

$$\frac{\operatorname{tg}(Q\vartheta + \phi) + \operatorname{tg}\tilde{\varphi}}{1 - \operatorname{tg}(Q\vartheta + \phi)\operatorname{tg}\tilde{\varphi}} = \left(1 + \frac{\delta Q}{Q}\right)\operatorname{tg}(Q\vartheta + \phi)$$

$$\operatorname{tg}\tilde{\varphi} = \frac{\delta Q}{Q}\,\frac{\operatorname{tg}(Q\vartheta + \phi)}{1 + \left(1 + \frac{\delta Q}{Q}\right)\operatorname{tg}^2(Q\vartheta + \phi)}$$

$$= \frac{\delta Q}{Q}\,\frac{\sin(Q\vartheta + \phi)\cos(Q\vartheta + \phi)}{1 + \frac{\delta Q}{Q}\sin^2(Q\vartheta + \phi)}$$

or, to second order in $\frac{\delta Q}{2Q}$, for $\frac{\delta Q}{Q} << 1$:

$$\operatorname{tg}\tilde{\varphi} = \frac{\delta Q}{2Q}\sin[2(Q\vartheta+\phi)] - \left(\frac{\delta Q}{2Q}\right)^2\left\{\sin[2(Q\vartheta+\phi)] - \frac{1}{2}\sin[4(Q\vartheta+\phi)]\right\}$$

$$= \frac{V}{2Q^2}\sin[2(Q\vartheta+\phi)] - \frac{V^2}{2Q^4}\sin[2(Q\vartheta+\phi)] + \frac{V^2}{8Q^4}\sin[4(Q\vartheta+\phi)] \tag{11.4}$$

Applying now the method of variation of constants, the perturbation Hamiltonian is

$$H = Vx^2 = V\frac{2a}{Q}\cos^2(Q\vartheta+\varphi) = \frac{Va}{Q}\Big(1+\cos[2(Q\vartheta+\varphi)]\Big) \tag{11.5}$$

From this and (9.4) the shift of frequency is

$$\delta Q = h_{110} = v_{110} = \frac{V}{Q} \quad \textit{in first approximation.}$$

Using (6.10) we find in second approximation

$$\delta Q = h_{110} = v_{110} - 4\frac{|v_{200}|^2}{2Q} = \frac{V}{Q} - \frac{V^2}{2Q^3}$$

in complete agreement with (11.1). The rapidly oscillating parts are found from (11.5) and (7.2) :

$$\tilde{\varphi} = \frac{1}{i}\left[\frac{v_{200}}{2Q}e^{i2(Q\vartheta+\phi)} - \frac{v_{020}}{2Q}e^{-i2(Q\vartheta+\phi)}\right] = \frac{V}{2Q^2}\sin[2(Q\vartheta+\phi)] ,$$

$$\tilde{a} = -\left[2\frac{v_{200}A}{2Q}e^{i2(Q\vartheta+\phi)} + \frac{2v_{020}}{2Q}A\,e^{-i2(Q\vartheta+\phi)}\right] = -\frac{VA}{Q^2}\cos[2(Q\vartheta+\phi)],$$

agreeing with the terms of first order in $\frac{V}{Q}$ following from (11.2) and (11.4). The conditions $\tilde{\varphi} \ll 1$, $\frac{\tilde{a}}{A} \ll 1$ obviously imply $\frac{V}{Q^2} \ll 1$ for the perturbation.

Having used this trivial example to illustrate the working of the method, we next investigate the example of a harmonic oscillator excited in a subresonance of 3rd order by a non-linear perturbation Hamiltonian with cosine dependence on ϑ :

$$H = V_3x^3 = \hat{V}\cos p\vartheta\, x^3,\ p \approx 3Q. \tag{11.6}$$

In the φ,a-representation, the following coefficients appear in the Hamiltonian

$$v_{30p} = \frac{\hat{V}}{2(2Q)^{3/2}} = v_{03,-p} = v_{30,-p} = v_{03,p}$$

$$v_{21p} = 3v_{30p} \quad = v_{12,-p} = v_{21,-p} = v_{12,p}$$

The rapid part of a is given by (7.2) :

$$\frac{\tilde{a}}{A} = -\,6v_{30p}\,A^{\frac{1}{2}} \left\{ \frac{\cos[(3Q+p)\vartheta + 3\phi]}{3Q+p} + \frac{\cos[(Q-p)\vartheta + \phi]}{Q-p} + \frac{\cos[(Q+p)\vartheta + \phi]}{Q+p} \right\}$$

from which

$$\left|\frac{\tilde{a}}{A}\right| \leq \left|6v_{30p}\,A^{\frac{1}{2}}\right| \left\{ \left|\frac{1}{3Q+p}\right| + \left|\frac{1}{Q-p}\right| + \left|\frac{1}{Q+p}\right| \right\}$$

$$\approx \left|6v_{30p}\,A^{\frac{1}{2}}\right| \left(\frac{1}{6Q} + \frac{1}{2Q} + \frac{1}{4Q}\right) = \frac{3\hat{V}A^{\frac{1}{2}}}{(2Q)^{3/2}Q}\;\frac{11}{12} \qquad (11.7)$$

$$= \frac{11}{48}\,\frac{3\hat{V}(\frac{2A}{Q})}{Q^2(\frac{2A}{Q})^{\frac{1}{2}}} = \frac{11}{48}\;\frac{\text{amplitude of non-linear force}}{\text{amplitude of linear force}}\;.$$

We compare this result with that for a perturbation consisting of δ-function kicks repeated after periods of 2π. In (11.6) we then write

$$V_3(\vartheta) = V\sum_{\nu=-\infty}^{\infty}\delta(\vartheta - 2\pi\nu) = \frac{V}{2\pi}\sum_{q=-\infty}^{+\infty} e^{-iq\vartheta}$$

The coefficients of the Hamiltonian become

$$v_{30q} = \frac{V}{2\pi(2Q)^{3/2}}\;,\; v_{21q} = \frac{3V}{2\pi(2Q)^{3/2}} \quad \text{etc.,}$$

and from (7.2) we find

$$\frac{\tilde{a}}{A} = -\,3v_{30p}\,A^{\frac{1}{2}} \left\{ \sum_{q\neq p} \frac{e^{-iq\vartheta}}{3Q-q}\,e^{i3(Q\vartheta+\phi)} + \sum_{q} \frac{e^{-iq\vartheta}}{Q-q}\,e^{i(Q\vartheta+\phi)} + \text{conj.complex} \right\}$$

The summations can be carried out in closed form, giving

$$\frac{\tilde{a}}{A} = -\frac{3VA^{\frac{1}{2}}}{2\pi(2Q)^{3/2}} \left\{ 2\pi \frac{\cos[3Q(\nu-\frac{1}{2})\ 2\pi + 3\phi]}{\sin 3Q\pi} - 2\frac{\cos[(3Q-p)\vartheta + 3\ \phi]}{3Q - p} + 2\pi \frac{\cos[Q(\nu-\frac{1}{2})\ 2\pi + \phi]}{\sin Q\pi} \right\}$$

$$= -\frac{3VA^{\frac{1}{2}}}{2\pi(2Q)^{3/2}} \left\{ 2\pi \frac{\cos\ [(3Q-p)(\nu-\frac{1}{2})2\pi + 3\ \phi]}{\sin\ (3Q-p)\pi} - 2\frac{\cos[(3Q - p)\vartheta + 3\ \phi]}{3Q - p} + 2\pi \frac{\cos[Q(\nu-\frac{1}{2})\ 2\pi + \phi]}{\sin Q\pi} \right\}$$

Herein ν = number of the period ϑ is lying in, i.e.

$$(\nu - 1)\ 2\pi < \vartheta < \nu\, 2\pi$$

Near a subresonance $3Q - p \rightarrow 0$ *and the above expression goes towards*

$$\frac{\tilde{a}}{A} = -\frac{3VA^{\frac{1}{2}}}{2\pi(2Q)^{3/2}} \left\{ 2[\vartheta - (\nu - \tfrac{1}{2})2\pi]\ \sin\ [(3Q - p)(\nu - \tfrac{1}{2})2\pi + 3\ \phi] + 2\pi \frac{\cos[Q(\nu-\frac{1}{2})2\pi + \phi]}{\sin Q\pi} \right\}$$

from which

$$\left|\frac{\tilde{a}}{A}\right| \leq \left|\frac{3VA^{\frac{1}{2}}}{2\pi(2Q)^{3/2}}\right| \left(2\pi + \left|\frac{2\pi}{\sin Q\pi}\right|\right) \rightarrow \left|\frac{3\frac{V}{2\pi}(\frac{2A}{Q})}{Q^2(\frac{2A}{Q})^{\frac{1}{2}}}\right| Q\frac{\pi}{2}\left(1 + \left|\frac{1}{\sin\frac{p\pi}{3}}\right|\right) =$$

$$= 1.08\pi Q\ \frac{\text{amplitude of exciting component of non-linear force}}{\text{amplitude of linear force}} \qquad (11.8)$$

(It has tacitly been assumed here that $\frac{p}{3}$ is not integral; if it were, the subresonance would coincide with a first order resonance). (11.8) turns out to be much larger than (11.7) (for the same ratio of exciting non-linear force to linear force), due to the large number of harmonics present in the excitation. Fixing a limit of $\frac{\tilde{a}}{A} < \frac{1}{10}$, the ratio "non-linear force to linear force" is allowed to reach $\sim \frac{4}{10}$ in the first case against $\sim \frac{1}{30Q}$ in the second case for the approximation to be equally good. In the second case the perturbations must be very small indeed for the theory to be applicable.

The separation of the perturbation Hamiltonian into a "low" and "high" frequency part encounters a difficulty if Q is about equally close to two rational numbers $\frac{p_1}{n_1}$ and $\frac{p_2}{n_2}$, as $Q = 6.30$, say, would be to $6\,1/3 = \frac{19}{3}$ and $6\,1/4 = \frac{25}{4}$. The procedure of section 9 requires the definition of one of the frequencies $Q - \frac{p_1}{n_1}$ or $Q - \frac{p_2}{n_2}$ as "low". If p_1 is chosen to be the exciting frequency in $\bar{H}$, the effect of p_2 appears in the rapidly varying part, and vice versa.

For the theory to hold, at least one of the choices must be compatible with the condition $\frac{\tilde{a}}{A} \ll 1$. If both subresonances are excited with about equal strength, the latter condition must be satisfied by either alternative. This means that the "slow" beating of the amplitude must be small too.

The equivalence of both alternatives can be seen as follows : Let

$$H = \bar{H} + \tilde{H}$$

be the decomposition of the perturbation Hamiltonian into a "low frequency" and a "high frequency" part. Inspection of section 7 and (6.7a) shows that $\tilde{H} = -\frac{d\sigma}{d\vartheta}$. Slow and rapid parts then obey the equations

$$\frac{d\phi}{d\vartheta} = \frac{\partial\bar{H}}{\partial A}, \qquad \frac{d\tilde{\varphi}}{d\vartheta} = \frac{\partial\tilde{H}}{\partial A}$$

$$\frac{dA}{d\vartheta} = -\frac{\partial\bar{H}}{\partial\phi}, \qquad \frac{\partial\tilde{a}}{d\vartheta} = -\frac{\partial\tilde{H}}{\partial\phi}$$

If in $\bar{H}$ the exciting frequency is p_1, and the effect of p_2 is preponderant in $\tilde{H}$, the interpretations of $\bar{H}$ and $\tilde{H}$ can be interchanged without affecting the result.

Taking the above example $Q \approx \frac{19}{3} \approx \frac{25}{4}$, we find for the two alternatives from (7.2) (most important terms only) :

$$\frac{\tilde{a}}{A} = -\frac{4\hat{V}_4R^3}{Q^2R}\frac{Q}{4Q - p_2}\frac{1}{8}\cos 4\left[(Q - \frac{p_2}{4})\,\vartheta + \phi\right], \quad \frac{p_1}{3} \approx Q$$

$$\frac{\tilde{a}}{A} = -\frac{3\hat{V}_3R^2}{Q^2R}\frac{Q}{3Q - p_1}\frac{1}{4}\cos 3\left[(Q - \frac{p_1}{3})\,\vartheta + \phi\right], \quad \frac{p_2}{4} \approx Q$$

where $\hat{V}_3$ and $\hat{V}_4$ are perturbation amplitudes defined by the perturbation Hamiltonian

$$H = \hat{V}_3 x^3 \cos p_1\vartheta + \hat{V}_4 x^4 \cos p_2\vartheta$$

and $R = \left(\frac{2A}{Q}\right)^{\frac{1}{2}}$ is the amplitude of the x-oscillation. In the more specified example $Q = 6.30 \approx \frac{19}{3} \approx \frac{25}{4}$ one obtains

$$\frac{\tilde{a}}{A} = -\frac{5}{8}\, Q\, \frac{4\hat{V}_4 R^3}{Q^2 R} \cos.., \qquad Q \approx \frac{p_1}{3}$$

$$\frac{\tilde{a}}{A} = \frac{5}{2}\, Q\, \frac{3\hat{V}_3 R^2}{Q^2} \cos.., \qquad Q \approx \frac{p_2}{4} \quad .$$

For these ratios to be $< \frac{1}{10}$, say, the ratios of the exciting non-linear forces to the linear force have to be smaller than about $\frac{1}{10Q}$.

The examples in this section show that the method of slowly varying amplitude and phase may be expected to give fairly good results for perturbations not exceeding the percent region. Richness in harmonics of the perturbation, interaction of several subresonances, and high Q-value tend to reduce the permissible perturbations even more.

However, in an essentially linear A.G. synchrotron, such as the CERN proton synchrotron, the perturbations are usually small enough for the present method to be useful.

12. Method of "variation of constants" for oscillations in two dimensions

Proceeding now to the more general case of betatron oscillations in the vertical (z-) direction as well as in the radial (x-) direction, we start from the Hamiltonian specified in (I.24)).

$$H(x, z, x', z', \vartheta) = H^{(0)} + H^{(1)} \quad . \qquad (12.1)$$

$$H^{(0)} = \frac{(x')^2 + (z')^2}{2} - n\frac{x^2 - z^2}{2}$$

is the Hamiltonian of the idealized "unperturbed" system obeying the equations of motion (see Appendix II).

$$x'' - n(\vartheta)x = 0 \tag{12.3}$$

$$z'' + n(\vartheta)z = 0 \ , \tag{12.4}$$

The unperturbed motions in the x- and z-directions are entirely decoupled. In both directions we introduce again phase and amplitude variables φ_1, φ_2, a_1, a_2 by

$$\begin{pmatrix} x \\ x' \end{pmatrix} = a_1^{\frac{1}{2}} \left[\begin{pmatrix} w_1(\vartheta) \\ w_2(\vartheta) \end{pmatrix} e^{i(Q_1\vartheta + \varphi_1)} + \begin{pmatrix} \overset{*}{w}_1(\vartheta) \\ \overset{*}{w}_2(\vartheta) \end{pmatrix} e^{-i(Q_1\vartheta + \varphi_1)} \right] \tag{12.5}$$

$$\begin{pmatrix} z \\ z' \end{pmatrix} = a_2^{\frac{1}{2}} \left[\begin{pmatrix} u_1(\vartheta) \\ u_2(\vartheta) \end{pmatrix} e^{i(Q_2\vartheta + \varphi_2)} + \begin{pmatrix} \overset{*}{u}_1(\vartheta) \\ \overset{*}{u}_2(\vartheta) \end{pmatrix} e^{-i(Q_2\vartheta + \varphi_2)} \right] \tag{12.6}$$

where w, u *are the Floquet factors of equations (12.3), (12.4) respectively*, and Q_1, Q_2 the "smooth motion "wave numbers per revolution (determined by the function $n(\vartheta)$). Following section 2, φ_1, φ_2, a_1, a_2 can be shown to obey Hamiltonian equations deriving from the perturbation Hamiltonian $H^{(1)}$, after substitution of (12.5), (12,6) for $\binom{x}{x'}$ and $\binom{z}{z'}$. $\binom{w_1}{w_2}$ and $\binom{u_1}{u_2}$ are supposed to be normalized according to

$$i \begin{vmatrix} w_1 & \overset{*}{w}_1 \\ w_2 & \overset{*}{w}_2 \end{vmatrix} = i \begin{vmatrix} u_1 & \overset{*}{u}_1 \\ u_2 & \overset{*}{u}_2 \end{vmatrix} = 1.$$

Restricting the perturbation Hamiltonian again to the potential energy we find

$$H^{(1)} = \sum_{k_1 k_2} V_{k_1 k_2} \, x^{k_1} z^{k_2} =$$

$$= \sum_{k_1, k_2} \sum_{\substack{\ell_1 + m_1 = k_1 \\ \ell_2 + m_2 = k_2}} \sum_{q} v_{\ell_1 m_1 \ell_2 m_2 q} \, a_1^{\frac{k_1}{2}} a_2^{\frac{k_2}{2}} \, e^{i[(\ell_1 - m_1)(Q_1\vartheta + \varphi_1) + (\ell_2 - m_2)(Q_2\vartheta + \varphi_2) - q\vartheta]} \tag{12.7}$$

where $v_{\ell_1 m_1 \ell_2 m_2 q}$ are defined by the Fourier expansion

$$\binom{\ell_1 + m_1}{m_1} \binom{\ell_2 + m_2}{m_2} w(\vartheta)^{\ell_1} \overset{*}{w}(\vartheta)^{m_1} u(\vartheta)^{\ell_2} \overset{*}{u}(\vartheta)^{m_2} V_{k_1 k_2}(\vartheta) = \sum_{q=-\infty}^{\infty} v_{\ell_1 m_1 \ell_2 m_2 q} e^{-iq\vartheta} \tag{12.8}$$

and

$$v_{m_1 \ell_1 m_2 \ell_2 (-q)} = v^*_{\ell_1 m_1 \ell_2 m_2 q} \quad , \tag{12.9}$$

similar to the one dimensional case.

We assume that no terms of order lower than 2 appear in the perturbation Hamiltonian (12.7), i.e. k_1+k_2 is > 2 for all terms. It was pointed out in section 8 that terms of the type $V_{10}x$ and $V_{01}z$ would cause a distortion of the "closed orbit", and that they can be removed by measuring displacements from the distorted closed orbit. The reasoning of section 8 remains valid in the two dimensional case.

So far everything is still rigorously correct. The next step is the restriction to the low frequency part of the perturbation Hamiltonian which gives the slow (and appreciable) variations of amplitude and phase. Inspection of (12.7) shows that in addition to the zero frequency terms with $\ell_1 = m_1$, $\ell_2 = m_2$, $q = 0$, zero or almost zero frequency terms appear if Q_1 and Q_2 satisfy a relation

$$n_1 Q_1 + n_2 Q_2 \approx p \quad , \; n_1, \; n_2, \; p \text{ integers.} \tag{12.10}$$

This is the well-known condition for the possiblity of a subresonance in two-dimensional oscillations, $|n_1| + |n_2| = n$ being called the "order" of the subresonance. In a Q_1, Q_2-diagram, the Q-values giving rise to subresonance lie on or near the rational straight lines $n_1 Q_1 + n_2 Q_2 = p$.

Explicitly the low frequency part of the Hamiltonian takes the form :

$$\begin{aligned}\bar{H} = {} & v_{11000} A_1 + v_{00110} A_2 + v_{22000} A_1^2 + v_{11110} A_1 A_2 + v_{00220} A_2^2 + \ldots \\ & + v_{|n_1| 0 |n_2| 0 p} A_1^{\left|\frac{n_1}{2}\right|} A_2^{\left|\frac{n_2}{2}\right|} e^{i[(n_1 Q_1 + n_2 Q_2 - p)\vartheta + n_1 \phi_1 + n_2 \phi_2]} + \ldots \\ & + \text{conj. complex.}\end{aligned} \tag{12.11}$$

Here capital letters ϕ_1, A_1, ϕ_2, A_2 have been introduced to denote the slowly varying part of the motion. Further it has been assumed that only one resonance relation (12.10) holds, i.e. that Q_1, Q_2 do not lie on or near an intersection of two or more resonance lines. Multiples of n_1, n_2, p *could of course be included in (12.11) without violating this condition.*

Extending the arguments of sections 5 and 6, the transition from $H^{(1)}$ of (12.7) to $\bar{H}$ of (12.11) can be interpreted as a canonical transformation to slowly varying variables ϕ_1, A_1, ϕ_2 A_2. However, we need not go into all this, if we are satisfied with the first approximation (12.11) to $\bar{H}$.

13. General discussion of the motion in two dimensions. Invariants

The general character of the motion is similar to that in one dimension, apart from effects of coupling.

If terms exciting a resonance are absent, both amplitudes are constant, whereas the frequencies are shifted by

$$\frac{d\phi_1}{d\vartheta} = \frac{\partial\bar{H}}{\partial A_1} = v_{11000} + 2v_{22000}\, A_1 + v_{11110}\, A_2 + \dots$$

$$\frac{d\phi_2}{d\vartheta} = \frac{\partial\bar{H}}{\partial A_2} = v_{00110} + v_{11110}\, A_1 + 2v_{00220}\, A_2 + \dots$$

the shifts depending in general on both amplitudes.

The presence of resonance terms causes the amplitudes to change. If only one resonance condition $n_1Q_1 + n_2Q_2 = p$ holds, two invariant relations between amplitudes and phases can be established (Beth (1910), Judd (1950), Hagedorn (1955), Sturrock (1955)). The first follows as in the one-dimensional case by calculating

$$\frac{d\bar{H}}{d\vartheta} = \frac{\partial\bar{H}}{\partial\vartheta} = Q_1\,\frac{\partial\bar{H}}{\partial\phi_1} + Q_2\,\frac{\partial\bar{H}}{\partial\phi_2} - \frac{p}{n_1+n_2}\left(\frac{\partial\bar{H}}{\partial\phi_1} + \frac{\partial\bar{H}}{\partial\phi_2}\right)$$

$$= -\,Q_1\,\frac{dA_1}{d\vartheta} - Q_2\,\frac{dA_2}{d\vartheta} + \frac{p}{n_1+n_2}\,\frac{d}{d\vartheta}\,(A_1 + A_2)$$

from which

$$\bar{H} + (Q_1 - \frac{p}{n_1+n_2})\, A_1 + (Q_2 - \frac{p}{n_1+n_2})\, A_2 = C_I. \qquad (13.1)$$

Here use has been made of the obvious property of the Hamiltonian (12.11)

$$\frac{1}{n_1}\,\frac{\partial\bar{H}}{\partial\phi_1} = \frac{1}{n_2}\,\frac{\partial\bar{H}}{\partial\phi_2}\,, \qquad (13.2)$$

and of the Hamiltonian equations of motion.

A second invariant is obtained immediately from the property of

(13.2) through the Hamiltonian equations :

$$\left(\frac{A_1}{n_1} - \frac{A_2}{n_2}\right) = C_{II} \ . \tag{13.3}$$

If either n_1 or n_2 is zero, we find A_1 = const. or A_2 = const. instead of (13.3).

The second invariant relation has a very interesting consequence : If n_1, n_2 have opposite signs, i.e. $|n_1|Q_1 - |n_2|Q_2 \approx p$, (13.3) shows that the amplitudes (which are proportional to $\sqrt{A}$) in x- and z- direction stay on a certain ellipse during the motion. Thus both amplitudes necessarily remain finite, and the resonance does not cause instability. In a "stable" resonance of this type always one of the amplitudes decreases while the other amplitude is increasing (a fact also exhibited directly by the Hamiltonian equations for A_1 and A_2). The particular case $|n_1|Q_1 - |n_2|Q_2 = 0$ is a "coupling resonance" where a growth of amplitude in one degree of freedom <u>must</u> be at the expense of the other one.

If n_1, n_2 have equal signs, the accessible ranges of amplitudes depend on the first invariant(13.1). The second invariant can be used to eliminate one of the amplitudes, A_2 say, from (13.1). Assuming that only one exciting term is present in $\bar{H}$ (i.e. only the terms written in full in (12.11); the same simplifying assumption was made in the detailed discussion of the one-dimensional motion in section 10), we arrive at

$$\begin{aligned} &(n_1Q_1 + n_2Q_2 - p + n_1 v_{11000} + n_2 v_{00110})A_1 \\ &+ n_1 \left(v_{22000} A_1^2 + v_{11110} A_1 \left[\frac{n_2}{n_1} A_1 - n_2 C_{II}\right] + v_{00220} \left[\frac{n_2}{n_1} A_1 - n_2 C_{II}\right]^2 \right) \\ &+ n_1 v_{|n_1|0|n_2|0p} A_1^{\left|\frac{n_1}{2}\right|} \left(\frac{n_2}{n_1} A_1 - n_2 C_{II}\right)^{\left|\frac{n_2}{2}\right|} e^{i[(n_1Q_1 + n_2Q_2 - p)\vartheta + n_1\phi_2 + n_2\phi_2]} \\ &+ \text{conj. complex} = C \ . \end{aligned} \tag{13.4}$$

This is a relation very similar to (10.1) which served to find the beating range of the amplitude of one dimensional oscillations. The general conclusions are correspondingly similar : The fourth order terms of zero-frequency in the Hamiltonian have the effect of limiting the beating ratio at large amplitudes for orders $|n_1| + |n_2| < 4$, and at small amplitudes for orders $|n_1| + |n_2| > 4$. The relevant parameter is the ratio

$$\kappa = \frac{\left[v_{22000} + v_{11110}\,\frac{n_2}{n_1} + v_{00220}\,\left(\frac{n_2}{n_1}\right)^2\right] A_{10}^2}{2\left|v_{|n_1|o|n_2|op}\right| \left(\frac{n_2}{n_1}\right)^{\left|\frac{n_2}{2}\right|} A_{10}^{\frac{|n_1|+|n_2|}{2}}}, \tag{13.5}$$

where A_{10} is the minimum amplitude of the beating. The higher the absolute value of κ, the more the beating is reduced.

If one of the numbers n_1, n_2 is zero, $n_2 = 0$ say, then $A_2 = \text{const.}$ and the motion is quasi-one-dimensional. In this case the invariant (13.1) can be used immediately to find the beating range; with $A_2 = \text{const.}$ it is of exactly the same form as that discussed in sections 9) and 10) for the one dimensional motion. The stabilizing parameter κ is obtained from (13.5) by putting $n_2 = 0$.

Resonance curves and maxima of beating ratios can also be found in a way similar to that of section 10, with the complication, however, that the value of C_{II} (which is determined by the ratio of the two initial amplitudes) enters as a further parameter. The details have been worked out for the particular case of a 3rd order subresonance $|n_1| + |n_2| = 3$ in a previous report (Hagedorn and Schoch (1957)). It was found that, for the same κ-value, the maximum beating ratio in two dimensional oscillations (defined as the ratio of the amplitude under consideration to the greater of the two initial amplitudes) exceeds the beating ratio in one dimensional oscillations at most by a factor of about 2. Thus the beating in one and two dimensional oscillations does not seem to differ in order of magnitude.

The invariant C_I was obtained by eliminating from $\frac{d\bar{H}}{d\vartheta} = \frac{\partial\bar{H}}{\partial\vartheta}$ the terms multiplied by exponentials using the Hamiltonian equations $\frac{dA_1}{d\vartheta} = -\frac{\partial\bar{H}}{\partial\phi_1}$ and $\frac{\partial A_2}{\partial\vartheta} = -\frac{\partial\bar{H}}{\partial\phi_2}$. The latter equations are not independent because of (13.2). If Q_1, Q_2 lie on the intersection of two rational lines in the (Q_1, Q_2)-plane, that is if simultaneously

$$\begin{aligned} n_1Q_1 + n_2Q_2 &= p \\ m_1Q_1 + m_2Q_2 &= q \end{aligned} \qquad n_1,\ n_2,\ p,\ m_1,\ m_2,\ q \ \text{ integers,}$$

it is still possible to obtain an invariant by the elimination of the terms with exponentials by means of the Hamiltonian equations for A_1, A_2 which now are independent. But there does not seem to be an obvious second invariant

$$\left[\begin{array}{ll} \alpha = \dfrac{A}{A_o} & \\ 2|h_{nop}|A_o^{\frac{n}{2}-1} & = \dfrac{2}{n}\,\Delta Q_e \\ h_{220}\,A_o & = \dfrac{2}{n}\,\Delta Q_e\,\kappa \\ Q + h_{110} - \dfrac{p}{n} & = \dfrac{2}{n}\,\Delta Q_e\,\xi \end{array}\right. \qquad (14.1)$$

the new Hamiltonian, written in the variables ψ, α, becomes

$$H = \frac{2}{n}\,\Delta Q_e\,\left[\xi(\vartheta)\alpha + \kappa\alpha^2 + \alpha^{\frac{n}{2}}\cos n\psi\right] \qquad (14.2)$$

If ξ is independent of ϑ, this H is the constant of the motion from which the phase plane paths have been derived in sections 9 and 10. Provided that $\xi(\vartheta)$ changes only very little within the period a particle takes to complete its phase plane cycle, H is still approximately invariant. In the phase plane the motion is, therefore, a sequence of gradually changing cycles each of which almost coincides with a cycle of the corresponding instantaneous system (with ξ kept constant). The change of cycles is restricted by Liouville's theorem, which requires that the phase plane area enclosed by a cycle remains constant. These properties of the motion define what is called an "adiabatic" change of the system.

The opposite extreme to an adiabatic change is a sudden jump of ξ. Then the phase pattern switches suddenly over to a different one, the moving point conserving its position in the phase plane during the jump, and continuing its motion on the new phase curve on which it finds itself after the jump. For this picture to apply, the switch over must be completed within a time which is small compared to the period of a cycle.

Applying these ideas to our present problem, a look at the phase diagrams in figs. 1-4 shows that a particle moving on a closed curve in the central region with ξ either below or above resonance will continue to move on a very similar closed curve after a sudden shift of ξ through resonance. If ξ is changed adiabatically through resonance, the particle can encounter phase paths going to infinity or phase paths forming separatrices. (A separatrix separates regions of closed curves encircling different fixed points).On both these kinds of paths the cycle period becomes infinitely long, so they to replace (13.3) which ceases to hold. The conclusions which can be drawn from the first invariant on the amplitude ranges in two dimensions are usually rather limited. Thus the behaviour of resonances at an intersection of several resonances lines remains a somewhat open question.

For more details on two-dimensional oscillations, reference is made to the reports by Hagedorn [1955, 1957].

14. Oscillations in systems with slowly changing parameters

So far, the parameters characterizing the system and the perturbations have been assumed to be constant in time. In reality, there will be small variations due to the energy oscillations of the particles connected with the acceleration mechanism, and due to unavoidable changes of the guiding field in the course of acceleration (e.g. by saturation of magnets). These changes are slow in comparison with the period of a betatron oscillation or even of a revolution.

Secular energy oscillations and changes of the field can be introduced into the original Hamiltonian (I.1) as secular variations of the momentum p and the vector potential A. Putting these variations into the perturbation part $H^{(1)}$ of the Hamiltonian, the coefficients $V_k(\vartheta)$ of the latter are no longer strictly periodic. They can, however, still be assumed in the form of Fourier series according to (4.4) with slowly varying coefficients $v_{\ell mq}(\vartheta)$.

The most important secular variations of the system are those altering the total frequency $Q + \frac{d\phi}{d\vartheta}$, because they can influence resonances. We consider, therefore, in particular a slowly changing coefficient h_{110} (see (9.2) and (9.4)) in the Hamiltonian

$$\bar{H} = h_{110}(\vartheta)\, A + h_{220}A^2 + 2\, h_{nop}|A^{\frac{n}{2}}| \cos n[(Q - \frac{p}{n})\vartheta + \phi + \beta].$$

If we introduce

$$\psi = (Q - \frac{p}{n})\, \vartheta + \phi + \beta$$

and furthermore use the abbreviations of section 10, cannot be passed adiabatically, and the application of the adiabatic theorem breaks down. It can, however, be applied to those parts of the phase plane which are not crossed by separatrices during the interval considered.

Starting at a state corresponding to phase diagrams like those for $\xi = -8$ on figures 1, 2, and 3, bottom part, and moving away from resonance (towards $\xi = -16$), central particles again approximately maintain their amplitudes, but particles moving in the islands round the excentric fixed points,are *trapped in these islands, as far as adiabatic changes go.* They move outward as the whole islands do, if ξ moves away from resonance. It was pointed out in section 10 that the oscillations corresponding to fixed points are synchronized with the exciting perturbation. The points trapped by fixed points therefore represent motions kept in synchronism with the perturbation ("locked" to the perturbation), in spite of the changing fundamental frequency. This trapping or lock-on process could play an important part in the loss of particles in synchrotrons, as had first been noticed by Adams and Hine [1953].

The instantaneous fixed points as functions of ξ are shown in figs.10 11 and 12 for $n = 2, 3$ and 4. The instable one is the intersection point of the separatrix; the maximum and minimum of α on the separatrix are also shown by dotted lines. *They mark upper and lower bounds for the amplitude of particles* trapped near a stable fixed point. The shaded region, therefore, represents the extension in α of the excentric islands.

It can be seen from fig. 5, section 10, that in fixed points $C = \text{const.}$ must be tangent to

$$C = \xi\alpha + \kappa\alpha^2 \pm \alpha^{\frac{n}{2}} . \qquad (14.3)$$

Thus they follow from

$$\frac{dC}{d\alpha} = \xi_f + 2\kappa\alpha_f \pm \frac{n}{2}\,\alpha_f^{\frac{n}{2}-1} = 0 \qquad (14.4)$$

A fixed point is stable or instable according to whether

$$\text{sign}\left(\frac{d^2C}{d\alpha^2}\right) = -\,\text{sign}\cos n\psi$$

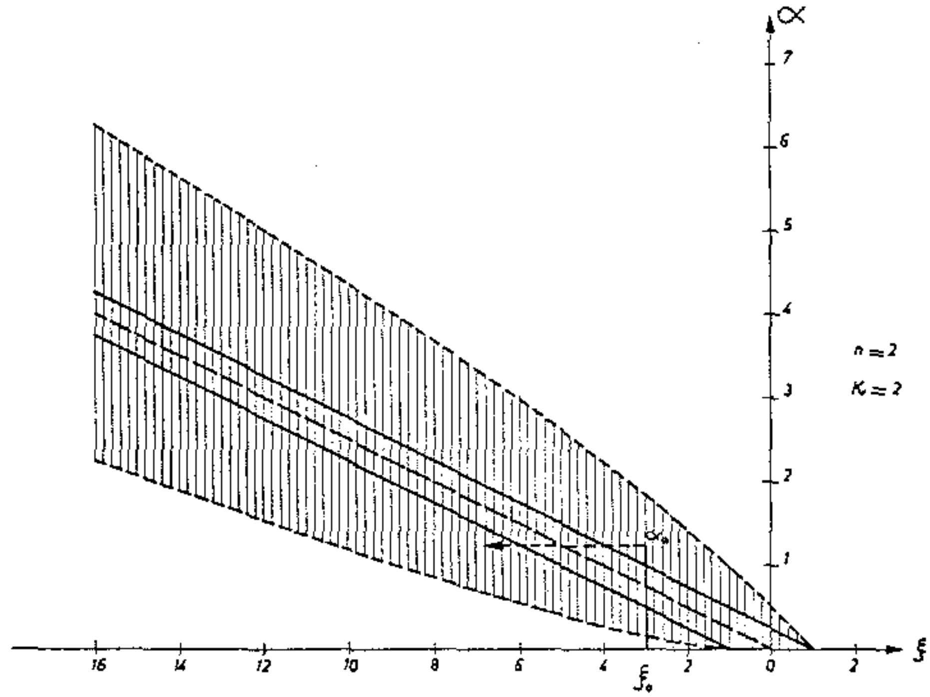

Fig. 10 Fixed points as function of ξ for 2nd order subresonance (upper full line : stable, lower full line : instable; shaded area : beating range of trapped particles).

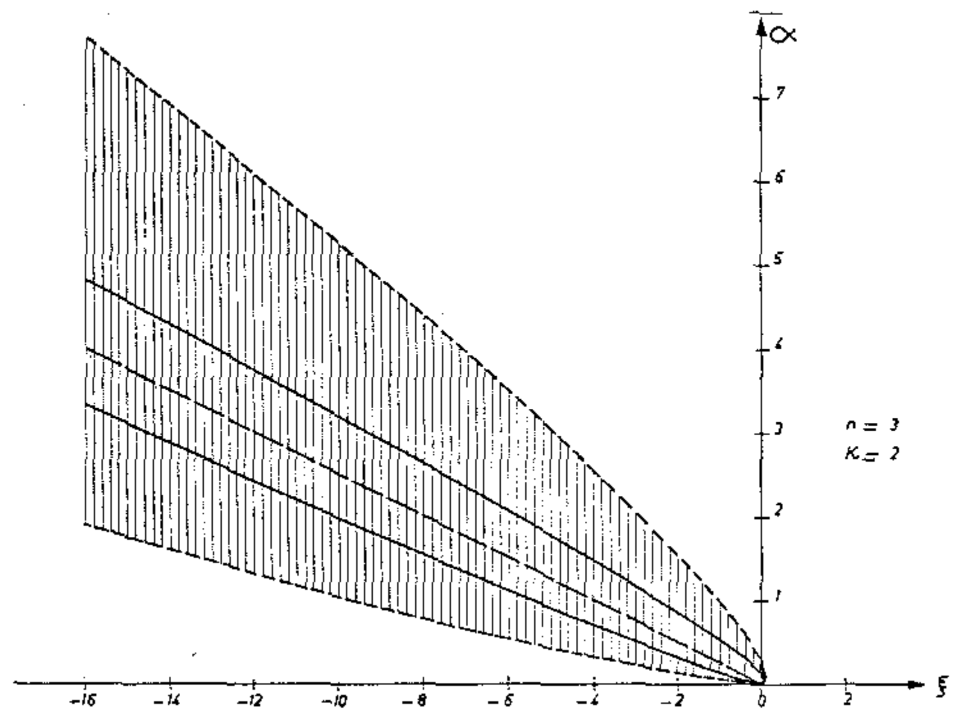

Fig. 11 Fixed points as function of ξ for 3rd order subresonance (upper full line : stable, lower full line : instable; shaded area : beating range of trapped particles).

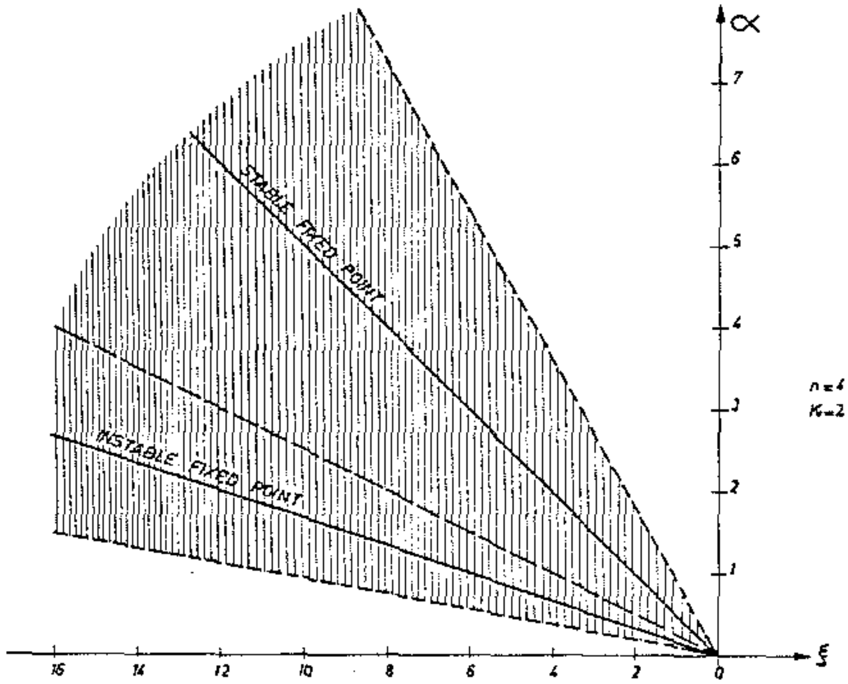

Fig. 12 Fixed points as function of ξ for 4th order subresonance.

or $$\operatorname{sign}\left(\frac{d^2C}{d\alpha^2}\right)_f = \operatorname{sign}\cos n\psi \; .$$

For $\kappa > 0$, the lower sign in (14.4) renders the stable fixed point, and the upper sign the unstable one, as can be inferred directly from fig. 5.

The separatrix is $C = C_s$, where from (14.3) and (14.4)

$$C_s = -\kappa\alpha_f^2 - (\tfrac{n}{2} - 1)\,\alpha_f^{\frac{n}{2}} \; .$$

Maximum and minimum α on the separatrix (dotted lines in figs.10,11,12) follow as intersections of $C = C_s$ with (14.3).

Suppose now that some range $o \ldots \alpha_o$ at some initial ξ_o is filled with particles. Then with ξ moving adiabatically to the left, as indicated in fig. 10, those of the particles in the shaded region whose phase falls into excentric islands will be dragged to higher amplitudes. A fraction of the particles initially above the shaded region will be captured later on into islands when the separatrix crosses their amplitude. Whether or not capture occurs depends on the phase in the moment a particle is passed by the separatrix :

Those near the instable fixed points of the separatrix have a good chance of finding themselves inside the central island(and below the shaded region) afterwards, because they almost do not move so that the conditions are rather those described above as a "jump". On the contrary, points being crossed by the separatrix sufficiently far off the unstable fixed points have a chance of finding themselves in one of the run-away islands after the crossing.

One would like to know which fraction of the particles, initially filling some given domain of phase space, is carried beyond a given limit of amplitude by a given change of ξ. This requires the knowledge of the evolution of the initial domain in phase space, the calculation of which is a difficult task, even though the Hamiltonian (14.2) looks relatively simple. The evolution obviously depends on the way ξ changes with ϑ; the faster this change, the more the motion will deviate from adiabatic behaviour.

There exists a threshold for the rate of change of ξ, above which a particle cannot be "trapped". It is clear for physical reasons, that enough time must be available for the exciting perturbation to change the energy of the system by the amount corresponding to the trapped motion. The equations of motion

$$\frac{d\psi}{d\vartheta} = \frac{\partial H}{\partial \alpha} = \frac{2}{n}\,\Delta Q_e\,\left(\xi + 2\kappa\alpha + \frac{n}{2}\,\alpha^{\frac{n}{2}-1}\cos n\psi\right)$$
$$\frac{d\alpha}{d\vartheta} = -\frac{\partial H}{\partial \psi} = \frac{2}{n}\,\Delta Q_e\,\left(n\alpha^{\frac{n}{2}}\sin n\psi\right), \qquad (14.5)$$

show that the fastest possible rate of rise of α is given by

$$\frac{d\alpha}{d\vartheta} = 2\Delta Q_e\,\alpha^{\frac{n}{2}} \quad . \qquad (14.6)$$

The ϑ-interval necessary for build-up from $\alpha_0 = 1$ to α follows by integration

$$\vartheta = \frac{\log\alpha}{2\Delta Q_e} \qquad n = 2$$
$$\vartheta = \frac{1 - \alpha^{1-\frac{n}{2}}}{(n-2)\Delta Q_e} \qquad n > 2 \qquad (14.7)$$

For the fastest rise to be maintained, $\sin n\psi = 1$ must be maintained, which leads to :

$$\frac{d\psi}{d\vartheta} = \frac{2}{n} \Delta Q_e \ (\xi + 2\kappa\alpha) = 0, \tag{14.8}$$

implying a certain program $\xi(\vartheta)$ by virtue of (14.7) :

$$\xi(\vartheta) = - 2\kappa \ e^{2\Delta Q_e \vartheta} \quad , \qquad n = 2$$
$$\xi(\vartheta) = \frac{-2\kappa}{[1 - (n-2)\Delta Q_e \vartheta]^{\frac{2}{n-2}}}, \ n > 2 \tag{14.9}$$

As a function of ξ, the time of fastest possible rise of α as given by (14.8) is midway between the lines for the stable and unstable instantaneous fixed points (figs. 10, 11 and 12, dashed line; note that this line at the same time marks the shift of the free oscillation frequency with amplitude, see (9.4).

In any motion, in which similar or steeper growth of α with $-\xi$ than defined by (14.8) occurs, the variation of ξ with ϑ must necessarily be slower than given by (14.9).

In particular, the adiabatic motion of the stable fixed points and of their vicinity shall be considered in some detail. The instantaneous fixed points were defined by

$$\frac{d\psi}{d\vartheta} = 0 \quad , \quad \frac{d\alpha}{d\vartheta} = 0,$$

giving, by means of the equations of motion (14.5),

$$\sin n\psi_f = 0, \qquad \xi + 2\kappa\alpha_f \pm \frac{n}{2} \alpha_f^{\frac{n}{2}-1} = 0, \tag{14.10}$$

in agreement with (14.4). With ξ changing adiabatically, there exists, of course, no longer a fixed point in the rigorous sense. Defining a quasi-fixed point still by the second of the last equations, the variation of α_f requires a slight shift of ψ_f to make

$$\frac{d\alpha_f}{d\vartheta} = 2\Delta Q_e \alpha_f^{\frac{n}{2}} \sin n\psi_f \neq 0. \tag{14.11}$$

If ξ changes slowly enough, $\cos n\psi_f = \pm 1$ still to a very good approximation, so that the second of the equations (14.10) is not noticeably affected. If ξ changes so rapidly that ψ_f and α_f are both affected considerably, the concept of a quasi-fixed point loses its meaning. Supposing, therefore $|\sin n\psi_f| << 1$, (14.11) is at least an order of magnitude smaller than the maximum rate (ΔQ_e being the same), so ξ has to vary also by an order of magnitude slower than would correspond to the threshold rate defined above, in order to allow fixed points to move adiabatically.

To investigate the motion of particles in the vicinity of quasi-fixed points, it is convenient to introduce the relative coordinates

$$\psi^* = \psi - \psi_f$$

$$\alpha^* = \alpha - \alpha_f .$$

This transformation can be derived as

$$\psi^* = \frac{\partial S(\psi,\alpha^*)}{\partial \alpha^*} ,$$

$$\alpha = \frac{\partial S(\psi,\alpha^*)}{\partial \psi} ,$$

from a generating function

$$S(\psi,\alpha^*) = (\psi - \psi_f)(\alpha_f + \alpha^*),$$

so that the Hamiltonian in the new coordinates is

$$H^* = H + \frac{\partial S}{\partial \vartheta} =$$

$$H(\psi_f,\alpha_f) + \left(\frac{\partial H}{\partial \psi}\right)_f \psi^* + \left(\frac{\partial H}{\partial \alpha}\right)_f \alpha^* + \frac{1}{2}\left(\frac{\partial^2 H}{\partial \psi^2}\right)_f \psi^{*2} + \frac{1}{2}\left(\frac{\partial^2 H}{\partial \alpha^2}\right)_f \alpha^{*2} + \left(\frac{\partial^2 H}{\partial \psi \partial \alpha}\right) \psi^* \alpha^* + ..$$

$$+ \frac{d\alpha_f}{d\vartheta}\psi^* - \frac{d\psi_f}{d\vartheta}(\alpha_f + \alpha^*)$$

where $H(\psi,\alpha)$ has been expanded in the neighbourhood of ψ_f, α_f up to terms of second order in ψ^*, α^*. As the ψ_f, α_f are supposed to obey the Hamiltonian equations

$$\frac{d\psi_f}{d\vartheta} = \left(\frac{\partial H}{\partial \alpha}\right)_f \quad , \qquad \frac{d\alpha_f}{d\vartheta} = -\left(\frac{\partial H}{\partial \psi}\right)_f \quad ,$$

H^* reduces to

$$H^* = H(\psi_f, \alpha_f) - \frac{d\psi_f}{d\vartheta}\,\alpha_f + \frac{1}{2} H_{\psi\psi}\,\psi^{*2} + \frac{1}{2} H_{\alpha\alpha}\,\alpha^{*2} + H_{\psi\alpha}\,\psi^*\alpha^*, \tag{14.12}$$

(in obvious notation for the derivatives of H). For constant $H_{\psi\psi}$ and $H_{\alpha\alpha}$ this would be the Hamiltonian of a harmonic oscillator. With our particular Hamiltonian (14.2)

$$\begin{aligned}
\frac{n}{2\Delta Q_e} H_{\psi\psi} &= -n^2\,\alpha_f^{\frac{n}{2}} \cos n\psi_f &&\approx n^2\,\alpha_f^{\frac{n}{2}} \\
\frac{n}{2\Delta Q_e} H_{\alpha\alpha} &= 2\kappa + \frac{n}{2}\left(\frac{n}{2}-1\right)\alpha_f^{\frac{n}{2}-2} \cos n\psi_f &&\approx 2\kappa - \frac{n}{2}\left(\frac{n}{2}-1\right)\alpha_f^{\frac{n}{2}-2} \\
\frac{n}{2\Delta Q_e} H_{\psi\alpha} &= -\frac{n^2}{2}\,\alpha_f^{\frac{n}{2}-1} \sin n\psi_f &&\approx 0
\end{aligned} \tag{14.13}$$

remembering that in a stable fixed point $\cos n\psi_f \approx -1$, $\sin n\psi_f \approx 0$. The term with $H_{\psi\alpha}$ in H^* can, therefore, be dropped.

For α_f varying adiabatically, the instantaneous phase paths are the curves

$$\frac{1}{2} H_{\psi\psi}\,\psi^{*2} + \frac{1}{2} H_{\alpha\alpha}\,\alpha^{*2} = C = \text{const.}$$

which are ellipses for $H_{\alpha\alpha} > 0$. (If they are not, the fixed point is not stable). The constant can be expressed by the maximum of ψ^* or α^* :

$$C = \frac{1}{2} H_{\psi\psi}\,\psi^{*2}_{max} = \frac{1}{2} H_{\alpha\alpha}\,\alpha^{*2}_{max} \;.$$

In the course of the adiabatic change the axes of the ellipse change, but Liouville' theorem requires constant area :

$$\psi^{*2}_{max}\left(\frac{H_{\psi\psi}}{H_{\alpha\alpha}}\right)^{\frac{1}{2}} = \alpha^{*2}_{max}\left(\frac{H_{\alpha\alpha}}{H_{\psi\psi}}\right)^{\frac{1}{2}} = \text{const.} \tag{14.14}$$

From (14.13)

$$\frac{H_{\alpha\alpha}}{H_{\psi\psi}} = \frac{2\kappa - \frac{n}{2}\left(\frac{n}{2} - 1\right)\alpha_f^{\frac{n}{2}-2}}{n^2\alpha_f^{\frac{n}{2}}}$$

This quantity (where it is positive) decreases with increasing α_f . Therefore ψ^*_{max} must decrease, α^*_{max} must increase with increasing α_f, a feature which is qualitatively exhibited by the phase diagrams in figs 1 to 3.

The frequency of the harmonic motion described by the Hamiltonian (14.12), i.e. the beating frequency, follows as

$$\omega = (H_{\psi\psi}\, H_{\alpha\alpha})^{\frac{1}{2}} = 2\Delta Q_e\, (2\kappa\alpha_f^{\frac{n}{2}} - \frac{n}{2}(\frac{n}{2} - 1)\, \alpha_f^{n-2})^{\frac{1}{2}},$$

increasing with α_f for $n \leqq 4$.

In the foregoing analysis, the motion about the fixed points has been treated in the linear approximation, valid for small beatings about the fixed points. For larger deviations, the beating becomes non-linear, the beating period approaching infinity as the separatrix is approached.

For the blow-up of amplitude by lock-in to happen, ξ has to move in the right sense, that is such that the fixed points move outward. (This sense depends on the sign of κ). If the sense is reversed, the amplitudesnear the fixed points go down with these. However, if the islands round the stable fixed points move outward, there must necessarily be other regions of phase space moving inward at the same time, and vice versa, because of Liouville's theorem. This phenomenon of "phase space displacement" has been discussed by Symon and Sessler (1956) in connection with the dynamics of particle acceleration, which shows an almost complete analogy with the adiabatic motion of trapped phase points discussed here. A particular consequence of this phase space displacement is that in crossing a resonance reversely to the lock-in sense, empty islands will be diving towards the center, displacing particles that occupy the central region to higher amplitudes. So in either sense of crossing resonance, increase and decrease of amplitude occurs, though a small fraction of phase space increasing amplitude by a large amount might be balanced by a large fraction decreasing amplitude by a small amount only, and vice versa.

After having dealt with an adiabatic change of ξ a rapid non-adiabatic change shall now be considered.

Integrating the second of the equations of motion (14.5) we obtain for the change of α caused by ξ sweeping through resonance :

$$\left.\begin{array}{c}\log \dfrac{\alpha_2}{\alpha_1} \\[2ex] \dfrac{\alpha_2^{\,1-\frac{n}{2}} - \alpha_1^{\,1-\frac{n}{2}}}{1-\frac{n}{2}}\end{array}\right\} = 2\Delta Q_e \int_{\vartheta_1}^{\vartheta_2} \sin n\psi \, d\vartheta \quad \text{for} \begin{cases} n = 2 \\ n > 2 \end{cases} \tag{14.15}$$

Because of the oscillatory character of the integrand, the main contribution to the integral comes from the neighbourhood of the point ϑ_0, where the phase ψ is stationary as a function of ϑ, that is where (see (14.5))

$$\left(\frac{d\,\psi}{d\,\vartheta}\right)_{\vartheta=\vartheta_0} = \frac{2}{n}\,\Delta Q_e \left(\xi_0 + 2\kappa\alpha_0 + \frac{n}{2}\,\alpha_0^{\frac{n}{2}-1} \cos n\psi_0\right) = 0,$$

ξ_0, α_0, ψ_0 denoting the values of ξ, α, ψ in ϑ_0. This condition is satisfied somewhere between the fixed point curves in figs. 10, 11 and 12.

In the neighbourhood of the stationary phase, we approximate ψ by its power series expansion up to the second order,

$$\psi = \psi_0 + \frac{1}{2}\,\psi_0'' \,(\vartheta - \vartheta_0)^2 \tag{14.16}$$

whereupon

$$\int_{\vartheta_1}^{\vartheta_2} \sin n\vartheta = \sin n\psi_0 \int_{\vartheta_1}^{\vartheta_2} \cos \frac{n\psi_0''}{2}\,(\vartheta - \vartheta_0)^2 \, d\vartheta + \cos n\psi_0 \int_{\vartheta_1}^{\vartheta_2} \sin \frac{n\psi_0''}{2}(\vartheta - \vartheta_0)^2 d\vartheta \tag{14.17}$$

appears expressed by Fresnel integrals. The Fresnel integrals approach the asymptotic value $\sqrt{\dfrac{\pi}{n|\psi_0''|}}$, as $\vartheta_2 \to \infty$ and $\vartheta_1 \to -\infty$, that is as ξ passes from a point a long way from subresonance on one side to a point a long way off on the opposite side. The asymptotic value is reached within about 20% if the intervals $|\vartheta_2 - \vartheta_0|$ and $|\vartheta_1 - \vartheta_0|$ are large enough to make

$$|n(\psi_2 - \psi_0)| = \left|\frac{n\psi_0'' \,(\vartheta - \vartheta_0)^2}{2}\right| \gtrsim \frac{100}{2\pi} \approx 16.$$

Within the range of ψ or ϑ given by this, ψ'' has to be fairly constant to justify the approximation (14.16, 14.17). Now from the first of equations (14.5)

$$\psi'' = \frac{d^2\psi}{d\vartheta^2} = \frac{2}{n}\,\Delta Q_e \left[\frac{d\xi}{d\vartheta} + \frac{d\alpha}{d\vartheta}\left(2\kappa + \frac{n}{2}\,(\frac{n}{2}-1)\alpha^{\frac{n}{2}-2}\cos n\psi\right) - \frac{d\psi}{d\vartheta}\,\frac{n^2}{2}\,\alpha^{\frac{n}{2}-1}\sin n\psi \right]$$

$$= \frac{2\Delta Q_e}{n}\left[\frac{d\xi}{d\vartheta} + 2\Delta Q_e \alpha^{\frac{n}{2}}\left(2\kappa + \frac{n}{2}(\frac{n}{2}-1)\alpha^{\frac{n}{2}-2}\cos n\psi\right) - \Delta Q_e n\alpha^{\frac{n}{2}-1}\left(\xi + 2\kappa\alpha + \frac{n}{2}\alpha^{\frac{n}{2}-1}\cos n\psi\right)\sin n\psi\right]$$

If, therefore $\left|\frac{d\xi}{d\vartheta}\right|$ is itself practically constant in the ϑ-interval of interest and much greater than all of the remainder in [ ], the condition of constant ψ'' is satisfied. It implies in particular

$$\left|\frac{d\xi}{d\vartheta}\right| >> \left|2\Delta Q_e \times 2\kappa\alpha^{\frac{n}{2}}\right|, \tag{14.18}$$

that is much greater than the limiting rate of change of ξ for lock-in motion, given by (14.6) and (14.8).

Combining (14.5) and (14.17), the asymptotic change of α is found from

$$\left.\begin{array}{c} \log \frac{\alpha_2}{\alpha_1} \\ \\ (\frac{n}{2}-1)[\alpha_1^{1-\frac{n}{2}} - \alpha_2^{1-\frac{n}{2}}] \end{array}\right\} = 2\Delta Q_e \sqrt{\frac{\pi}{n|\psi_0''|}}(\sin n\psi_0 \pm \cos n\psi_0) \tag{14.19}$$

$$\approx \left(\frac{2\Delta Q_e 2\pi}{\left|\frac{d\xi}{d\vartheta}\right|}\right)^{\frac{1}{2}} \sin\,(n\psi_0 \pm \frac{\pi}{4}) \qquad \text{for} \qquad \begin{cases} n = 2 \\ n > 2, \end{cases}$$

the upper and lower signs holding respectively for $\frac{d\xi}{d\vartheta} \gtrless 0$. α increases or decreases according to the value of the stationary phase ψ_0, which depends on the initial conditions. There is no influence of the direction of sweep in this case of very fast sweep. For $\left|\frac{d\xi}{d\vartheta}\right| \to \infty$ (sudden jump) the change of α tends to zero, in agreement with earlier statements.

The maximum change of α, occurring for an appropriate phase, is

$$\left.\begin{array}{c} \left|\log \frac{\alpha_2}{\alpha_1}\right|_{max} \\ \\ (\frac{n}{2}-1)\left|\alpha_1^{1-\frac{n}{2}} - \alpha_2^{1-\frac{n}{2}}\right|_{max} \end{array}\right\} \approx \left(\frac{4\Delta Q_e^2\ 2\pi}{n\left|\frac{dQ}{d\vartheta}\right|}\right)^{\frac{1}{2}} = \frac{2\pi\ 2\Delta Q_e}{(n|\Delta Q_{rev}|)^{\frac{1}{2}}}\ ,\quad \begin{cases} n = 2 \\ n > 2 \end{cases} \tag{14.20}$$

where $\frac{d\xi}{d\vartheta}$ has been expressed by $\frac{dQ}{d\vartheta}$ or by ΔQ_{rev}, the change ΔQ_{rev} of the frequency per revolution 2π.

Expressing the condition (14.18) for the validity of the foregoing analysis in terms of ΔQ_{rev}, we find

$$\left|\frac{\Delta Q_{rev}}{2\Delta Q_e}\right| >> \left|\frac{8\pi\kappa\Lambda Q_e}{n}\alpha^{\frac{n}{2}}\right| = \left|2\pi\Delta Q_s\alpha^{\frac{n}{2}}\right| . \qquad (14.21)$$

ΔQ_s is the non-linear shift of frequency for the reference amplitude $\alpha = 1$ as can be seen from (14.1) and (10.6). Assuming for example $\alpha \approx 1, \Delta Q_s \approx 0.1$, ΔQ_{rev} must be many times the stopband width for the approximation to be valid. (14.21) at the same time implies that the r.h.s. of (14.19) or (14.20) is much smaller than 1 in practice, and the percentage change of amplitude in one passage therefore small.

Even a fast crossing of a subresonance might lead to appreciable build-up of the amplitude if repeated a great number of times. Such can be the case due to the oscillations of the Q-value produced by the phase oscillation of a particle under the influence of the accelerating radio frequency. An estimate of this kind of build-up may be made on a statistical basis, assuming random phases ψ_0 in the individual crossings. After N crossings, the r.m.s. expectation values are $(\frac{N}{2})^{\frac{1}{2}}$ times the maximum values for single passage in (14.20).

15. Conclusions on tolerances for imperfections of A.G. synchrotrons with particular reference to the CERN Proton Synchrotron.

In this final section the theory is applied to examine working conditions of the CERN Proton Synchrotron as far as betatron oscillations are concerned.

The basic design parameters : radius r_0 of curvature in magnet sectors, length of straight sections, number M of magnet periods, number Q of betatron oscillations per revolution, and aperture of the vacuum chamber are considered as given. Their choice is narrowed down by requirements on maximum particle energy, optimum acceptance (transversal and longitudinal),

* A remark may be added to this section to clarify a point regarding the well known adiabatic theorem by Boltzmann and Ehrenfest. This theorem states that a harmonic oscillator with varying frequency Q (but no excitation forces) $\frac{a}{Q}$=const. (a being the square-of-amplitude variable used in this report). From (14.5), however, A = const. follows in this case. The discrepancy is removed by taking the rapidly varying part $\tilde{a}$ of a into account.

pulse repetition rate, accomodation of accelerating, injection, and ejection devices and targets to the following set of values.

radius of curvature in magnet sectors	r_0 = 7008 cm
mean radius	r_m = 10000 cm
number of magnet periods	M = 50
number of straight sections	2M = 100
number of betatrons oscillations horizontally and vertically	$Q_1 = Q_2 = 6\ ^1/_4$

The mean radius corresponds to a fraction $\frac{2\pi r_m - 2\pi r_0}{2\pi r_m} = 30\%$ of the total length of the orbit consisting of straight sections. The straight sections are arranged in the middle of each "F-sector" (radially focussing, n negative) and of each "D-sector" (radially defocussing, n positive), leading to the structure with two straight sections per period shown in fig. 15 b. The 100 straight sections are not of equal length; 20 among them have been made 3.0 m. long each against 1.6 m. for each of the remaining 80, both kinds of course uniformly distributed round the machine. By this a superperiod has been created in the azimuthal structure, now composed of S = 10 superperiods of 5 magnet periods each. The effects of this superstructure are very weak, so we will here assume 2 M = 100 straight sections of equal length, summing up to the same total length. Each of these would then be 1.88 m. long.

The field index necessary to produce the desired values of Q_1, Q_2 in the ideal structure, defined by fig. 15 b and the equations of motion(12.3), (12.4) is found to be

$$|n| = 282.4 \quad .$$

Within the number of decimal places given, the value is the same for the structure with short and long straight sections, and for the structure with equal straight sections having the same total length. A graph of Q vs. n in the vicinity of n = 282 given in fig.13 shows the sensitivity of Q to changes of n.

It must be noted that the definition of n adopted here is

$$n = -\frac{r_0}{B_0}\frac{\partial B_z}{\partial x},$$

referring to r_0 and not to r_m. Furthermore the azimuthal position variable

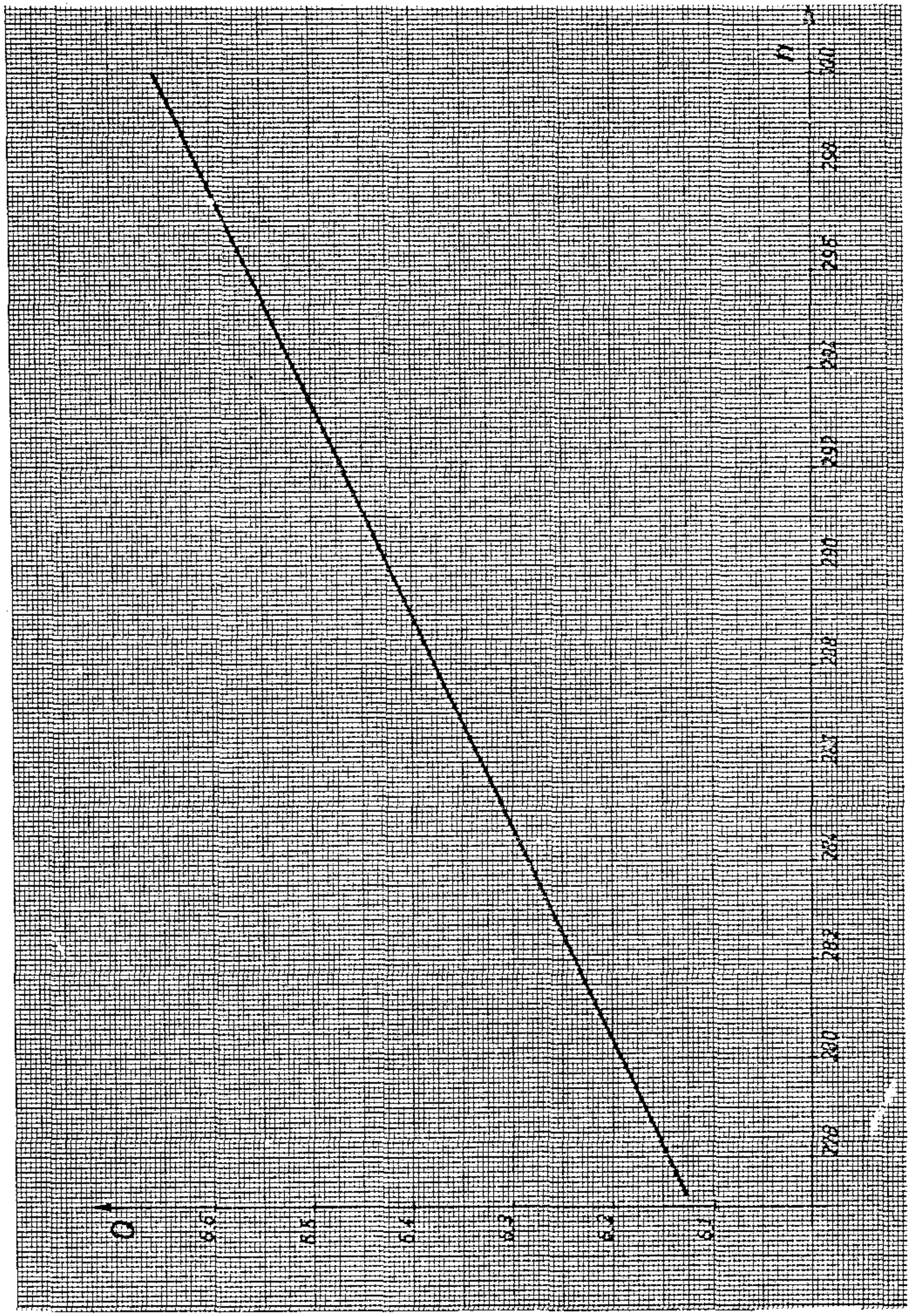

Fig. 13 Q vs. n for the basic structure of the CERN PS in the neighbourhood of the nominal value n = 282.4.

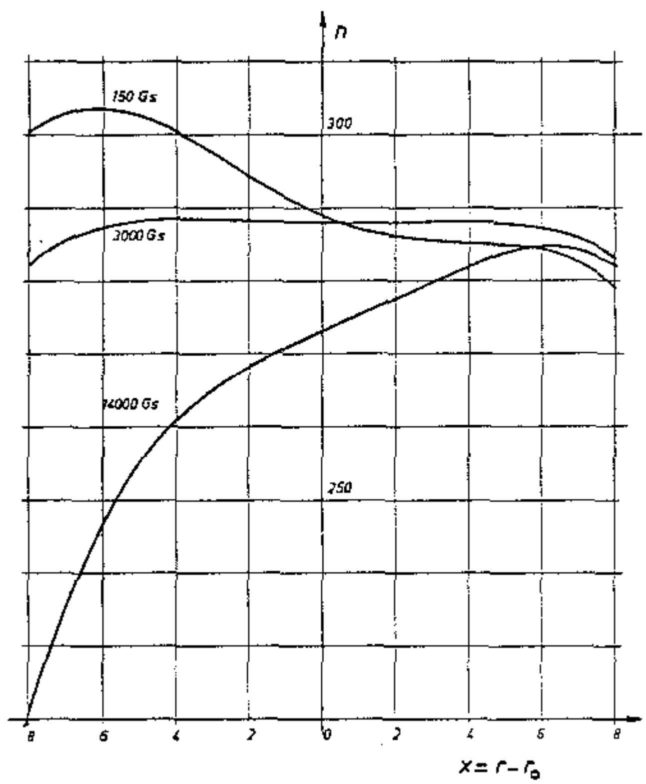

Fig. 14 n vs. $x = r - r_o$ in the CERN PS magnet gaps at injection, medium and maximum field. The curve drawn for injection includes effects of remanent field and eddy currents in the vacuum chamber.

ϑ has to be understood as length along the nominal equilibrium orbit divided by the mean radius r_m. As a consequence of these definitions, in the equations of motion of the basic system (3.1) or (12.3)(12.4), $n(\vartheta)$ has to be replaced by $\frac{r_m^2}{r_0^2} n(\vartheta)$. See Appendix I and II for further explanations.

The problem now is to know the influence of systematic and accidental construction deficiencies on the stability of betatron oscillations, on which the survival of the beam depends. The basic parameters chosen properly, instability is exclusively due to resonance with exciting circumferential periodicities in the structure.

The first condition is therefore that no undesirable periodicity is systematically present in the structure on account of its design. The undesirable harmonics p are given by (12.10)

$$n_1 Q_1 + n_2 Q_2 = p,$$

where n_1, n_2 are positive integers. (If n_1, n_2 have opposite signs, the resonance does not produce instability, see section 13). The linear resonances of order $|n_1| + |n_2| = 1$ and 2 have already been avoided by choosing $Q_1 = Q_2 \approx$ integer + $\frac{1}{4}$. *Checking on possible non-linear resonances inside the range* $Q_1 = 6 \dots 6\,1/2$, $Q_2 = 6 \dots 6\,1/2$, one finds the following list of harmonic numbers p :

$\lvert n_1\rvert + \lvert n_2\rvert$	1	2	3	4	5	6	7	8	9	10	11
p			19	25	31	37	43	49	55	62	67
						38		<u>50</u>	56	63	68
								51	57	64	69
											<u>70</u>

The harmonics of the structure being multiples of 50, or taking account of the superperiod, multiples of 10, only the underlined numbers could occur as exciting harmonics. The corresponding resonances would be of order 8 at least. As resonances in general become less disturbing with increasing order, the choice of Q with respect to the number of periods and superperiods can be considered as rather good. Below order 8, the proper exciting harmonics can be produced by irregularities only.

The influences disturbing the ideal structure, on which the n-value given above is based, are analyzed and incorporated in the equations of motion in Appendix I. They appear in the perturbation Hamiltonian $H^{(1)}$ whose coefficients contain the information we need .

First we have to mention a number of systematic influences leaving undisturbed the regular periodicity of the structure but affecting the radius of the equilibrium orbit and the Q-values :

(i) Deviations of the n-value from the designed value due to profile errors, due to remanent fields and eddy currents at low fields, and due to saturation effects at high fields. The behaviour of n in the CERN PS magnet as a function of the radial distance x from the designed equilibrium orbit is shown in fig.14 for injection, medium and maximum fields. In the medium field region, practically extending from 2.000 to 10.000 Gauss, n is fairly constant over the radial width of the vacuum chamber ($|x| \lessapprox 7$ cm), whereas non-linearities, as well as deviations from the medium field value on the central orbit $x = 0$, occur at low and high fields.

(ii) Fringing fields around magnet ends : their effects can be discussed in terms of an equivalent length of azimuthally constant field continuing that inside a magnet sector. At the ends facing field free sections the magnets are effectively lengthened by an amount $\Delta\ell$ given in the table below for the CERN PS at various fields B_0 according to measurements (CERN PS magnet group [1957]).

$\Delta\ell$ increases towards the wide gap side, roughly linearly, and $\frac{d\Delta\ell}{dx}$ is also given in the table. This variation of effective length with x diminishes the focusing power of the fringing field, resulting in an effective length for the field gradient of only $\Delta\ell_G = \Delta\ell - \frac{r_0}{n}\frac{d\Delta\ell}{dx}$.

Effective length of fringing fields at magnet ends

B_0 =	150	3000	14000	Gauss
$\Delta\ell$ =	0,2	7.1	5	cm
$\frac{d\Delta\ell}{dx}$ =	± 0.15	± 0.15	± 0.20	cm^{-1} for ± n
$\Delta\ell_G$ =	- 3.7	3.3	0	cm

(iii) Focusing action on radial oscillations by unbalanced centrifugal force. Its effect is to make $Q_1 \neq Q_2$, which would otherwise be equal for the basic structure with zero average of n considered here.

(iv) Deviations of particle momentum from the equilibrium value. In the CERN PS the momentum range corresponding to stable phase oscillations is about $\frac{\delta p}{p_0} = \pm 3.10^{-3}$ (at injection). Additional momentum deviations can be caused by errors in the frequency of the accelerating voltage, carrying the total possibly up to $\frac{\delta p}{p_0} = \pm 10^{-2}$ near injection.

In general these influences are strong enough to require compensation. Unbalanced centrifugal force can be corrected for and Q_1 made equal to Q_2 by making D-sectors slightly longer than F-sectors (by 1.8 cm in the CERN PS). Compensating the bending effect of fringing fields by making the magnet sectors shorter by $\Delta\ell$ at each end, the loss in focusing power must be corrected by an increase of the n-value (by about 1.6% in the CERN PS). This correction can be incorporated in the pole profile for medium fields. Variable corrections are, however, required at low and high fields as the magnet n-value as well as the effective lengths depend on the field level. They are effected by poleface windings and by quadrupole lenses. Two sets of 10 quadrupole lenses each in appropriate arrangement allow to move the Q-values in the (Q_1, Q_2)-plane in arbitrary directions. Corrections of non-linearities will be done partly also by pole face windings, partly by sextupole and octupole lenses arranged in two sets of 10 again for each type (see Appendix V and Appendix VI for more details).

As to the influence of momentum deviations there is a slight effect on the Q-values, because the focusing is weaker for higher momentum.

Additional effects on the Q-values appear in the case of non-linearities (i.e. if n or the effective lengths are not constant across the aperture). The latter effects are due to the fact that a particle deviating in momentum moves on an orbit of different radius and hence sees different n-values.

A study made by Adams and Hine [1956] and based on an earlier, less perfect magnet model demonstrated the importance of the displacement of the Q-values in the $(Q_1 Q_2)$-plane accompanying radial displacements of the equilibrium orbit. How the Q-values may be expected to move with orbit displacements in the final CERN PS, with no corrections of non-linearities in operation, is shown tentatively by the figs. 20...22 (Appendix V).

Q_1, Q_2 should stay away from resonances, first of all from integral values. It is clear from the diagrams that correction devices are indispensable at high fields for achieving this. There is some chance that the CERN PS could work without corrections at low fields.

Resonances become significant only by the presence of the following unsystematic influences :

(i) Irregularities of the guide field B_0, caused by misaligned magnets azimuthal fluctuations of magnetic properties and of eddy currents in metallic parts (e.g. the vacuum chamber).

(ii) Azimuthal irregularities of the field index n

(iii) Azimuthal fluctuations of the radial derivatives $\frac{dn}{dx}$, $\frac{d^2n}{dx^2}$, etc of n (which characterize non-linearities).

The first type of irregularities produces deformations of the equilibrium (or "closed") orbit. Not knowing the irregularities in an individual machine, statistical estimates of the closed orbit deformations have been made to obtain tolerance figures for B_0-irregularities (see e.g. CERN reports by Adams and Hine [1953d, 1954], Lüders [1953b, 1955, 1956]). A treatment of this question by the methods of the present report is contained in Appendix V. Assuming an admissible closed orbit deviation of 2 cm, $Q_1 = Q_2 = 6.25$, and no non-linearities, the error tolerances found are $\frac{\delta B}{B_0} \lesssim 10^{-3}$ for field fluctuations, $\xi \lesssim 0.05$ cm for misalignments of magnets, and $\Delta\epsilon \lesssim 0.007$ for internal twists of magnet units, the figures being r.m.s. fluctuations from magnet unit to magnet unit.

The closed orbit distortions are influenced by non-linearities, in particular because of the shift of the Q-values with momentum produced by such non-linearities and mentioned already above. The problem is discussed in some detail in Appendix V, with the result that the effect of momentum shift depends rather sensitively on the exact nature of the non-linearities. Zero quadratic non-linearity plus a cubic non-linearity with appropriate azimuthal Fourier harmonics (see V(41)) might well serve to reduce the danger of integral resonances by repelling the Q-values automatically from these with increasing closed orbit displacement.

<u>Azimuthal irregularities of n, $\frac{dn}{dx}$, $\frac{d^2n}{dx^2}$</u>, .., containing the relevant harmonics, may excite subresonances of orders 2, 3, 4.. respectively. To give an idea of how large such irregularities must be to produce disturbing effects we shall calculate the number of revolutions necessary for building up the amplitudes of betatron oscillations by a given factor. The figures obtained have to be judged by comparison with the rate of "adiabatic damping" of betatrons oscillations. The adiabatic damping, caused by the process of acceleration of the particles(and, therefore, not included in the theory presented in the preceding sections) makes the amplitudes of free betatron oscillations decrease according to a law

$$\frac{R(t_2)}{R(t_1)} = \left[\frac{B(t_2)}{B(t_1)}\right]^{-\frac{1}{2}} ,$$

B(t) being the guide field at time t. Thus a damping by a factor of $\sqrt{2}$ occurs during an increase of B by a factor of 2, which corresponds to an increase in *particle energy by between 4 at non-relativistic energies, and 2 at relativistic* energies. In the CERN P.S., the numbers of revolutions required for a $\sqrt{2}$ damping are

$$N \approx \frac{200 - 50}{0.05} = 3{\cdot}10^3 \text{ around injection energy}$$

$$N \approx \frac{24000 - 12000}{0.05} = 2.4\cdot 10^5 \text{ around top energy,}$$

as the particles gain energy at a constant rate of 0.05 MeV/turn, are injected at 50 MeV, and accelerated up to about 24000 MeV. Therefore, resonance phenomena become important if they can build up amplitudes by a factor of $\sqrt{2}$ in a number of revolutions smaller than those above.

The minimum number of revolutions necessary for a given build-up is obtained if conditions of perfect resonance are maintained, and follows from (14.7). For a ratio $\frac{R}{R_o} = (\frac{\alpha}{\alpha_o})^{\frac{1}{2}} = \sqrt{2}$ of final to initial amplitude we find

$$N = \frac{\vartheta}{2\pi} = \begin{cases} \dfrac{\log 2}{2\pi . 2\Delta Q_e} = \dfrac{0.0552}{\Delta Q_e} & m = 2 \\[2ex] \dfrac{1 - 2^{-\frac{1}{2}}}{2\pi\Delta Q_e} = \dfrac{0.0466}{\Delta Q_e} & m = 3 \\[2ex] \dfrac{1 - 2^{-1}}{2\pi\cdot 2\Delta Q_e} = \dfrac{0.0398}{\Delta Q_e} & m = 4 \\[2ex] \dfrac{1 - 2^{-3/2}}{2\pi\cdot 3\Delta Q_e} = \dfrac{0.0343}{\Delta Q_e} & m = 5 \end{cases} \quad \text{for order}$$

ΔQ_e *is the "excitation width" defined in (10.7) or (14.1) :*

$$\Delta Q_e = \begin{cases} 2|h_{20q}| = \dfrac{1}{2Q}\left(\dfrac{r_m}{r_o}\right)^2 \left[\delta n_q + \left(c\,\dfrac{dn}{dx}\right)_q + \dfrac{1}{2}\left(c^2\,\dfrac{d^2n}{dx^2}\right)_q + \ldots\right] \\[2ex] 3\left(\dfrac{Q}{2}\right)^{\frac{1}{2}} R_o|h_{30q}| = \dfrac{1}{8Q}\left(\dfrac{r_m}{r_o}\right)^2 R_o\left[\left(\dfrac{dn}{dx}\right)_q + \left(c\,\dfrac{d^2n}{dx^2}\right)_q + \ldots\right] \\[2ex] 4\,\dfrac{Q}{2}\,R_o^2\,|h_{40q}| = \dfrac{1}{24Q}\left(\dfrac{r_m}{r_o}\right)^2 \dfrac{R_o^2}{2}\left[\left(\dfrac{d^2n}{dx^2}\right)_q + \ldots\right] \\[2ex] 5\left(\dfrac{Q}{2}\right)^{3/2} R_o^3|h_{50q}| = \dfrac{1}{64Q}\left(\dfrac{r_m}{r_o}\right)^2 \dfrac{R_o^3}{3!}\left[\left(\dfrac{d^3n}{dx^3}\right)_q + \ldots\right] \end{cases} \quad \text{for} \begin{cases} m = 2 \\ m = 3 \\ m = 4 \\ m = 5 \end{cases}$$

In writing this the canonical amplitude variable

$$A = \left(\frac{R}{2w_o}\right)^2 \approx \frac{Q}{2} R^2$$

has been expressed in terms of the actual amplitude R, and the Hamiltonian Fourier coefficients h_{moq} have been expressed in terms of the synchrotron parameters (see Appendix VI, first approximation). The suffix q indicates the angular harmonic of the quantity in question. Unlike the earlier notation, the order of resonance is denoted by m here to avoid confusion with the field index n.

The statistical r.m.s. expectation value of the Fourier component δn_q can be calculated from

$$< | \delta n_q |^2 > = \frac{\sum_{k=1}^{N} \delta n^2 (\vartheta_k)}{U^2} = \frac{< \delta n^2 >}{U} ,$$

if the values $\delta n(\vartheta_k)$ in different magnet units (or sub-units like blocks) are statistically independent. U is the number of units (or sub-units) . ($<|\delta n_q|^2>$ is independent of the subdivision of the magnet chosen for calculating the average under the assumed statistical conditions).

In the CERN PS, the relative r.m.s. block to block fluctuation $\frac{<\delta n^2>^{\frac{1}{2}}}{n}$ seems to be a few times 10^{-4} according to first experience in magnet production.

Characterizing the fluctuations of $\frac{dn}{dx}$, $\frac{d^2n}{dx^2}$,.. by $\frac{1}{n} <(R_1 \frac{dn}{dx})^2>^{\frac{1}{2}}$, $\frac{1}{n} <(\frac{1}{2} R_1^2 \frac{d^2n}{dx^2})^2>^{\frac{1}{2}}$ (i.e. by the fluctuations of n due to azimuthal random irregularities of non-linearities at distance R_1 from the central orbit), they also seem to amount to a few times 10^{-4} at $R_1 = 4$ cm (with no correlation to the n-fluctuations at $R_1 = 0$). The number of blocks being U = 1000, the r.m.s. expectation values of the Fourier components become

$$\left(\frac{\delta n}{n}\right)_q \approx \frac{R_1}{n} \left(\frac{dn}{dx}\right)_q \approx \frac{1}{2} \frac{R_1^2}{n} \left(\frac{d^2n}{dx^2}\right)_q \approx \frac{10^{-4}...10^{-3}}{\sqrt{1000}} \approx 3.10^{-6} ... 3.10^{-5} .$$

Using the figure 3.10^{-6} (corresponding to 10^{-4} block to block fluctuation) and disregarding the contributions of closed orbit distortions $c(\vartheta)$ first, we find the following excitation widths and minimum numbers of revolutions for $\sqrt{2}$ build-up :

res. order	m	ΔQ_e	N
	2	$1.47\cdot 10^{-4}$	380
	3	$1.84\cdot 10^{-5}$	2500
	4	$3.1\cdot 10^{-6}$	12900
	5	$0.57\cdot 10^{-6}$	60000

For orders $m > 2$, ΔQ_e and N depend on the initial amplitude R_o. The above figures hold for $R_o = \frac{R_1}{2} = 2$ cm, this being the kind of beam radius one would like to keep inside the vacuum chamber. Although the perturbations assumed are quite small and put severe requirements on the uniformity of magnet production, the build-up could outweigh the adiabatic damping for resonance orders 2 and 3 already at the beginning of the acceleration cycle, and for order 4 and 5 near the end of the cycle.

Equally important or even larger contributions to the excitation width may be caused by closed orbit distortions, if appreciable systematic non-linearities exceeding the fluctuations are present. The important harmonics are $q \approx 2Q, 3Q, \ldots$ for $m = 2, 3, 4, \ldots$, i.e. $q = 13, 19, 25$ in the CERN PS where $6.0 < Q < 6.5$. ($q = 12, 18, 24$ coincide with first order resonances which are likely to be excited stronger anyway). Considering that the harmonics of $c(\vartheta)$ are strongest around 6, and that the number of lens periods is 10 (the non-linearities produced by lenses thus containing the harmonics 10, 20, 30, ...) the following excitation terms might become disturbing :

m = 2

$$\left(c\,\frac{dn}{dx}\right)_{13} = \sum_s c_{13-s}\left(\frac{dn}{dx}\right)_s = c_{13}\left(\frac{dn}{dx}\right)_0 + c_3\left(\frac{dn}{dx}\right)_{10} + c_{-7}\left(\frac{dn}{dx}\right)_{20} + \ldots$$

$$\frac{1}{2}\left(c^2\frac{d^2n}{dx^2}\right)_{13} = (c_0c_{13}+c_6c_7+..)\left(\frac{d^2n}{dx^2}\right)_0 + (c_0c_3+c_6c_{-3}+..)\left(\frac{d^2n}{dx^2}\right)_{10} + (c_0c_{-7}+..)\left(\frac{d^2n}{dx^2}\right)_{20}$$

m = 3

$$\left(c\frac{d^2n}{dx^2}\right)_{19} = c_{19}\left(\frac{d^2n}{dx^2}\right)_0 + c_9\left(\frac{d^2n}{dx^2}\right)_{10} + c_{-1}\left(\frac{d^2n}{dx^2}\right)_{20} + \ldots$$

Only the terms likely to be the largest have been written down explicitly. If c is the radial deviation of the perturbed closed orbit, the zero harmonic (average displacement) c_0 may be 1 cm or more due to momentum deviations. Assuming the higher harmonics c_q being produced by random displacements of magnet units, and corresponding to alignment tolerances ξ such as required in a preceding paragraph, their expectation values are (see Appendix V)

$$c_q \approx 0.7\,\xi \frac{Q^2}{Q^2 - q^2} \approx 0.04 \frac{Q^2}{Q^2 - q^2}\ [\mathrm{cm}],$$

Specifically :

c_0	$c_{\pm 1}$	$c_{\pm 3}$	$c_{\pm 6}$	$c_{\pm 7}$	$c_{\pm 9}$	$c_{\pm 13}$	$c_{\pm 19}$	
(1)	0.04	0.05	0.5	0.16	0.04	0.012	0.005	cm

The order of magnitude of

$$\left(\frac{dn}{dx}\right)_{0,\ 10,\ 20\ldots},\ \left(\frac{d^2n}{dx^2}\right)_{0,\ 10,\ 20\ldots}$$

can be estimated on various assumptions :

(i) no lenses in operation, and therefore only magnet non-linearities effective;

(ii) lenses operated to compensate magnet non-linearities (only $\frac{d^2n}{dx^2}$ need be considered in this case, as $\frac{dn}{dx}$ is planned to be corrected to zero by pole face windings on the magnets in the CERN PS);

(iii) lenses operated to produce certain desired non-linearities, e.g.

(iiia) use of sextupole lenses for suppressing non-linear dependence of closed orbit displacement on particle momentum (see eq. (V.20); that this effect may be undesirable at transition energy was pointed out by Johnsen [1956];

(iiib) use of octupole lenses for reducing closed orbit distortions and resonance effects (see (V.41), and this section further below. Details on the harmonic analysis of non-linearities and lenses are found in Appendices III, V, and VI).

For these various cases the figures obtained for the single contributions to the excitation widths are put down in the table below :

Subresonance excitation by closed orbit distortions in the CERN PS

Order m = 2

$$\Delta Q_e = \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2 \left[c_{13}\left(\frac{dn}{dx}\right)_0 + c_3\left(\frac{dn}{dx}\right)_{10} + c_{-7}\left(\frac{dn}{dx}\right)_{20} + (c_0c_{13} + c_6c_7)\left(\frac{d^2n}{dx^2}\right)_0 + (c_0c_3 + c_6c_{-3})\left(\frac{d^2n}{dx^2}\right)_{10} + c_0c_{-7}\left(\frac{d^2n}{dx^2}\right)_{20} + \dots \right]$$

		$c_{13}\left(\frac{dn}{dx}\right)_0$	$c_3\left(\frac{dn}{dx}\right)_{10}$	$c_{-7}\left(\frac{dn}{dx}\right)_{20}$	c_0c_{13}	c_6c_7	c_0c_3	c_6c_{-3}	c_0c_{-7}
Case (i)	150 Gauss	$3\cdot10^{-3}$							
	3000 "	$2\cdot10^{-4}$							
	14000 "	$4\cdot10^{-3}$							
Case (ii)	150 Gauss						$5\cdot10^{-4}$	$2.5\cdot10^{-4}$	1.10^{-3}
	3000 "						$1\cdot10^{-4}$	$0.6\cdot10^{-4}$	$0.8\cdot10^{-3}$
	14000 "						$4\cdot10^{-5}$	$2\cdot10^{-5}$	$2\cdot10^{-4}$
Case (iiia)		$3\cdot10^{-4}$	$1\cdot10^{-3}$	$3\cdot10^{-3}$					
Case (iiib)					$0.7\cdot10^{-4}$	$5\cdot10^{-4}$	$4\cdot10^{-4}$	2.10^{-4}	$2\cdot10^{-3}$

Order m = 3

$$\Delta Q_e = \frac{1}{8Q}\left(\frac{r_m}{r_o}\right)^2 R_o \left[c_{19}\left(\frac{d^2n}{dx^2}\right)_0 + c_9\left(\frac{d^2n}{dx^2}\right)_{10} + c_{-1}\left(\frac{d^2n}{dx^2}\right)_{20} + \dots \right]$$

		$c_{19}\left(\frac{d^2n}{dx^2}\right)_0$	$c_9\left(\frac{d^2n}{dx^2}\right)_{10}$	$c_{-1}\left(\frac{d^2n}{dx^2}\right)_{20}$
Case (i)		0	0	0
Case (ii)	150 Gauss		$2\cdot10^{-4}$	4.10^{-4}
	3000 "		$5\cdot10^{-5}$	$1\cdot10^{-4}$
	14000 "		$1.5\cdot10^{-5}$	$3\cdot10^{-3}$
Case (iiia)		0	0	0
Case (iiib)		$1.5\cdot10^{-5}$	$1.5\cdot10^{-4}$	$2.5\cdot10^{-4}$

In these figures, the lens harmonics in cases (ii), (iiia), (iiib) have been calculated for the lens arrangement in the CERN PS. In cases (iiia) and (iiib) the non-linearities have been assumed to be exclusively due to lenses; they would not change in order of magnitude if the magnet non-linearities had been taken into account.

The figures show that the excitation due to distortion of the closed orbit may easily exceed that due to a 10^{-4} block to block fluctuation by an order of magnitude. Clearly also, this excitation is strongly favoured by the harmonics introduced by the lens structure. In particular the 3rd order resonance excitation terms written down are entirely produced by the lens structure*.

Build-up of amplitudes by resonance can be stabilized by a cubic non-linearity of appropriate amount if the machine properties remain constant (Section 10). In order to keep beating of amplitudes within a factor of $\sqrt{2}$, the parameter κ defined by (10.10) (essentially the ratio of the cubic part of the focussing force to the excitation force) has to be $\gtrsim 10$ ... 20. Using (10.7) and expressing $A_0 = \frac{Q}{2} R_0^2$ by the amplitude R_0, (10.10) can be written for a m-th order resonance

$$\kappa = \frac{mQ}{4} \frac{h_{220} R_0^2}{\Delta Q_e}, \qquad (15.1)$$

where h_{220} is in terms of machine parameters (see Appendix VI) :

$$h_{220} \simeq v_{220} \approx - \frac{1}{16\,Q^2} \left(\frac{r_m}{r_0}\right)^2 \left[\left(\frac{d^2n}{dx^2}\right)_0 + 4\rho_1 \left(\frac{d^2n}{dx^2}\right)_M + .. \right]$$

For CERN PS data, an initial amplitude $R_0 = 2$ cm, and $|\kappa| = 20$, one obtains for

order	$m = 2$	3
excitation width $\Delta Q_e =$	$3 \cdot 10^{-3}$	$3 \cdot 10^{-4}$

$$\left| \left(\frac{r_m}{r_0}\right)^2 \left[\left(\frac{d^2n}{dx^2}\right)_0 + 0.67 \left(\frac{d^2n}{dx^2}\right)_M \right] \right| \approx \frac{64Q|\kappa|}{mR_0^2} \Delta Q_e \simeq 10^3 \Delta Q_e = 3 \qquad \frac{10^3}{3} \Delta Q_e = 0.1$$

If this cubic non-linearity would be produced by the basic magnet structure, $\frac{1}{n}\frac{d^2n}{dx^2}$ required in the magnets would be given by (see Appendix VI)

* See erratum

$$\frac{4Q^2}{n}\frac{d^2n}{dx^2} = -\left(\frac{r_m}{r_o}\right)^2\left[\left(\frac{d^2n}{dx^2}\right)_o + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M\right]$$

or

$$\left|\frac{d^2n}{dx^2}\right| = \begin{matrix}5.3\\0.18\end{matrix} \quad \text{for} \quad m = \begin{matrix}2\\3\end{matrix} \; ,$$

corresponding to a change of n of 2.6%, resp. 0.09% per cm in radial direction. The first figure, for suppressing the second order resonances, would mean a considerable non-linearity whereas the figure necessary for 3rd order resonances is small (in fact close to the values that must be expected anyway).

It was mentioned in a preceding paragraph that it was desirable to choose the cubic non-linearity such as to minimize closed orbit distortions. Favourable values for the harmonics, given in (V.41), would have to satisfy the conditions

$$\left(\frac{r_m}{r_o}\right)^2\left[\left(\frac{d^2n}{dx^2}\right)_o + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M\right] = -2\left(\frac{r_m}{r_o}\right)^2\left(\frac{d^2n}{dx^2}\right)_o \approx -0.075 \qquad (V.41)$$

These conditions would roughly meet the requirements for suppression of the third subresonance at the same time. They can be adjusted by octupole lenses (see Appendix V). The sign of $\frac{d^2n}{dx^2}$ is prescribed by these conditions; it does not matter for the suppression of subresonances.

The foregoing conditions for stabilization of non-linear resonances hold strictly for one dimensional (radial) oscillations only. In the general case we have to consider several possibilities of subresonance, e.g. corresponding to the relations

$$3Q_1 = 19, \; 3Q_2 = 19, \; 2Q_1 + Q_2 = 19, \; Q_1 + 2Q_2 = 19$$

for third order. The κ-values differ for these resonances. For equal excitation they are found to be of equal order of magnitude from (13.5), giving also the same order of magnitude for the beating factors (see section 13).

The intensity of excitation may also differ for the various subresonance lines. In particular, the resonances

$$Q_1 + Q_2 = 13$$
$$3Q_2 = 19$$
$$2Q_1 + Q_2 = 19$$

are excited only by perturbations destroying the plane of symmetry $z = 0$, i.e. by twisted magnets or vertical deformations of the closed orbit (see Appendix VI).

If a 3rd order subresonance would be provoked solely by closed orbit distortion, the adjustment of κ would not be completely free, because both stabilizing and exciting non-linearity then are proportional to $\frac{d^2n}{dx^2}$. In this case the requirement of a given value of κ sets a maximum tolerance for closed orbit distortion.

Under real working conditions, the Q-values are not fixed but undergo small changes (i) due to the momentum oscillations which accompany the acceleration process, (ii) due to changes of the magnetic field caused by a number of influences that have already been mentioned and are difficult to eliminate completely. Therefore, dynamical crossing of subresonances can happen. This may upset non-linear stabilization by particles getting locked to resonance if the sweep of frequency is in the right sense (i.e. balances amplitude dependent non-linear detuning).

The fastest rate of change of Q, at which lock-in is possible was shown in (14. 6) and (14.8) to be

$$\frac{d\xi}{d\vartheta} = -2\kappa \frac{d\alpha}{d\vartheta} = -4\kappa \Delta Q_e \alpha^{\frac{m}{2}}$$

or, expressed as change of Q per revolution

$$\left(\Delta Q_{rev}\right)_{limit} = -\frac{16\pi\kappa\Delta Q_e^2}{m} \alpha^{\frac{m}{2}} = \frac{\pi}{4} \frac{\Delta Q_e}{Q} \alpha^{\frac{m}{2}} R_o^2 \left(\frac{r_m}{r_o}\right)^2 \left[\left(\frac{d^2n}{dx^2}\right)_o + 4\rho_1 \left(\frac{d^2n}{dx^2}\right)_M\right] \approx 4\cdot 10^{-2} \alpha^{\frac{3}{2}} \Delta Q_e$$

where κ has been introduced from (15.1), and the numerical value at the end refers to 3rd order subresonance, the non-linearity (V.41) used above, and $R_o = 2$ cm.

First we examine the rates of change of Q due to momentum oscillations. Assuming sinusoidal oscillations of Q, the maximum rate would be

$$\left(\frac{dQ}{d\vartheta}\right)_{max} = Q_{ph} \hat{\Delta Q}$$

if $\hat{\Delta Q}$ is the amplitude of the oscillation and Q_{ph} the number of phase oscillation per revolution. Phase oscillations approaching the limit of phase stability are, it is true, no longer sinusoidal, and their frequency depends on amplitude. But in the greater portion of the phase stable region the foregoing formula is good enough for our present purpose. The average rate of change, over a complete half oscillation is

$$< \frac{dQ}{d\vartheta} > = \frac{Q_{ph}}{\pi} \; 2\Delta\hat{Q} = \frac{2}{\pi} \left(\frac{dQ}{d\vartheta}\right)_{max} .$$

Instead of $\frac{dQ}{d\vartheta}$ we shall use the change of Q per revolution.

$$\Delta Q_{rev} = 2\pi \frac{dQ}{d\vartheta} .$$

Q_{ph} and $\frac{(\Delta Q_{rev})_{max}}{\Delta\hat{Q}}$ are given below for a few values of the field B.

There is a limiting $\Delta\hat{Q}$, above which lock-in is impossible as ΔQ_{rev} becomes $>(\Delta Q_{rev})_{limit}$. $(\Delta\hat{Q})_{limit}$ is also given below in units of ΔQ_e.

Lock-in limits for Q-amplitudes :

B	=	150	1200	2640	4500	12000	Gauss
Q_{ph}	=	$5.3\cdot10^{-2}$	$1.8\cdot10^{-3}$	0	$7.1\cdot10^{-4}$	$5.2\cdot10^{-4}$	$\frac{osc.}{rev}$
$\frac{(\Delta Q_{rev})_{max}}{\Delta\hat{Q}}$	=	$3.3\cdot10^{-1}$	$1.1\cdot10^{-2}$	0	$4.5\cdot10^{-3}$	$3.3\cdot10^{-3}$	
$\frac{\Delta\hat{Q}_{limit}}{\Delta Q_e}$	=	0.11	3.4	∞	8	11	

We saw before that $\Delta Q_e = 10^{-4}$ (or even somewhat more) might be a likely value for the excitation width. So lock-in throughout the oscillation *could not take place for Q-amplitudes greater than*

	150	1200	2640	4500	12000	Gauss
$\Delta\hat{Q}_{limit} \approx$	$1\cdot10^{-5}$	$3\cdot10^{-4}$	∞	$8\cdot10^{-4}$	$1\cdot10^{-3}$	

The corresponding increase of amplitude would be smaller than given by

$\alpha = \left(\frac{R^2}{R_o^2}\right) \approx$	1.007	1.2	-	1.6	1.7

Apart from the vicinity of "transition" field (2640 Gauss), where the frequency of phase oscillation passes through zero, and to which we return later, *the Q-amplitudes compatible with continuous lock-in are quite small* and so is the corresponding change of betatron amplitudes. One should not forget that we are considering limiting rates, and unless the change of Q is properly programmed, the rates of change have really to be much smaller for lock-in to occur. On the other hand, the limiting lock-in rate increases with $\alpha = (\frac{R}{R_o})^2$; this has been neglected here as the approximate figures above would be affected only by factors < 2.

In order to show what meaning the limiting Q-amplitudes have in terms of the phase stable momentum range, we note that

$$\frac{\Delta p}{p_o} = - \frac{\Delta Q}{Q} = - 0.16\ \Delta Q$$

if non-linearities are very small; for the CERN P.S. without non-linear corrections,

$$\frac{\Delta p}{p_o} = \begin{array}{lll} + 0.016\ \Delta Q & & 150 \\ - 0.55\ \ \Delta Q & \text{at}\ \ B = & 3000\ \text{Gauss} \\ - 0.013\ \Delta Q & & 14000 \end{array}$$

would hold. Upon introducing $\Delta\hat{Q}_{limit}$ here, a very small fraction of the phase stable momentum range is found to be affected by possible lock-in.

The region about transition requires separate consideration. To give an idea of orders of magnitude we ask within which region the lock-in range $\Delta\hat{Q}_{limit}$ is $> 10^{-2}$, corresponding to a momentum range $|\frac{\Delta p}{p_o}| \gtrsim 1.5 \cdot 10^{-3}$. It turns out that this is the case within an interval of about 20 Gauss about the transition field, which is traversed in a time corresponding to about 700 revolutions. Now the minimum number of revolutions required for a $\sqrt{2}$ build-up is about 500, as follows from the figures on page 63 . Thus during transition lock-in blow-up in one sweep (the interval considered covers less than one phase oscillation) might just become possible.

Examining the effects of repeated crossing of a subresonance a lock-in build-up should be almost reversible, if the resonance is crossed adiabatically. As for non-adiabatic crossing (which would apply to the majority of the particles captured), a formula (14.20) derived for a speed by far exceeding the lock-in rate becomes valid if (see (14.21))

$$\Delta Q_{rev} >> (\Delta Q_{rev})_{limit} \approx 4.10^{-2}\ \Delta Q_e \qquad (m = 3).$$

For $\Delta Q_e = 10^{-4}$, this seems to be satisfied for $\Delta Q_{rev} > 10^{-4}$. We put down the maximum growth ratios $[\alpha = (\frac{R}{R_o})^2]_{max}$ for one crossing through $m = 3$, calculated for a Q-amplitude $\hat{\Delta Q} \approx 10^{-2}$ (equivalent to $|\frac{\Delta p}{p_0}| \approx 1.5 \cdot 10^{-3}$) :

B	=	150	1200	2640	4500	12000	Gauss
ΔQ_{rev}	$\approx$	$3.3 \cdot 10^{-3}$	$1.1 \cdot 10^{-4}$	0	$4.5 \cdot 10^{-5}$	$3.3 \cdot 10^{-5}$	
α_{max}	$\approx$	1.05	1.35	-	-	-	

These figures are not negligible, because there are several hundred phase oscillations between injection and transition, so the effect can accumulate to appreciable amplitudes (perhaps 2 or 3 times the initial amplitude) by statistical addition. Actual growth ratios may even be larger, as the rates of sweep are rather intermediate between the adiabatic and extremely non-adiabatic case.

Apart from phase oscillations, variations of the field index n with B_0 produce also changes of Q. Most of these changes will be compensated for by the correction devices (pole face windings and lenses). Remaining effects may well be slow enough for lock-in resonances. There are, however, the changes due to phase oscillations always superposed.

The figures used above for the speed of crossing a resonance may be much smaller in such cases where a resonance line in the Q_1, Q_2-diagram is swept over at an almost grazing angle, the likelihood of lock-in then being correspondingly greater.

Summarizing, one may conclude from the foregoing estimates that probable imperfections of the CERN PS suffice for appreciable excitation of subresonances, decreasing in strength with increasing order of the resonance ($\sqrt{2}$ build-up potentially in tens, hundreds or thousands of revolutions respectively for order 2, 3, 4). It must be recalled that the estimates were based on construction tolerances which are difficult to improve.

The suppression of subresonance effects by suitable non-linearities would, for the second order, require more cubic non-linearity than obtainable from octupole lenses in the CERN PS, whereas the amount required for third and higher order resonances would easily be available. But the suppression becomes partially ineffective because the Q-values are in general varying in time. On the other hand, apart from the vicinity of "transition" these variations seem to be so fast for most of the particles, that the effects of one single crossing of a 3rd order subresonance are weak. However, there may be accumulation in repeated crossings. So 3rd order resonances might still become a serious danger under unfortunate constellations of conditions. 4th and higher order resonances are less likely to be disturbing.

With regard to these facts, the most efficient way to avoid trouble would be to keep the Q-values safely away from 1st, 2nd and 3rd order resonance lines. This in turn requires the Q-values not to change too strongly with momentum, which necessitates careful control of non-linearities.

As to the agents exciting subresonances, possible improvements are mainly on closed orbit distortions. They may be pushed beyond construction tolerances by controlled compensating perturbations. (The feasibility of such compensations depends on the extent of information available on closed orbit behaviour). Apart from this, again careful control of the Q-values is essential also for closed orbit straightening. The major part of the excitation by closed orbit distortions would, of course, be eliminated by not using non-linear lenses.

In order to control the Q-values to the accuracy required, it is necessary to eliminate most of the quadratic non-linearity, at the high and low end of the field range at least, for reducing dependence on momentum deviation (compare figs. 20, 21, 22). In the CERN PS this will be achieved mainly by pole face windings on the magnets. In this way the increase of width of the half-integral stopbands, which would be produced by sextupole lenses, is avoided.

Whether the remaining cubic non-linearity can be left uncorrected depends on the radial displacements of closed orbits occuring as a result of momentum errors. This is a question of precision of the frequency program for the accelerating radio frequency. If the closed orbits stay within ± 2 cms from the center of the vacuum chamber, it may well turn out that the octupole lenses need not be operated.

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* * *

Appendix I. Equations of motion in alternating gradient synchrotrons

For the perturbation method used in this paper we need the equations of motion in Hamiltonian form. In cylindrical coordinates r, ϑ, z for the position of the particle, the Hamiltonian of a particle in an electromagnetic field is

$$H = c\left[(p_r - \frac{e}{c}A_r)^2 + (\frac{p_\vartheta}{r} - \frac{e}{c}A_\vartheta)^2 + (p_z - \frac{e}{c}A_\vartheta)^2 + (m_r c)^2\right]^{\frac{1}{2}},$$

where A_r, A_ϑ, A_z the components of the vector potential, m_r the rest mass of the particle, c the velocity of light, and absolute electrostatic units are used. The canonical momenta p_r, p_ϑ, p_z are defined by the first set of Hamilton's equations :

$$\frac{dr}{dt} = \dot{r} = \frac{\partial H}{\partial p_r} = \frac{c^2}{H}(p_r - \frac{e}{c}A_r)$$

$$\frac{d\vartheta}{dt} = \dot{\vartheta} = \frac{\partial H}{p_\vartheta} = \frac{c^2}{H}\frac{1}{r}(\frac{p_\vartheta}{r} - \frac{e}{c}A_\vartheta)$$

$$\frac{dz}{dt} = \dot{z} = \frac{\partial H}{p_z} = \frac{c^2}{H}(p_z - \frac{e}{c}A_z) \quad .$$

The value of H equals the energy of the particle (including rest energy), $\frac{H}{c^2}$ is its instantaneous relativistic mass.

As we restrict ourselves to motions without acceleration, A_r, A_ϑ, A_z do not depend on time, so that $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$ expresses the fact that energy is constant for a particle moving in a magnetic field.

Introducing ϑ instead of t as an independent variable, the equations of motion are still of Hamiltonian form, with a different Hamiltonian however, (see Bell [1955], also Whittaker's Analytical Dynamics [1937]), which can be inferred from the following set of equations :

$$\frac{dr}{d\vartheta} = r' = \frac{\dot{r}}{\dot{\vartheta}} = \frac{\frac{\partial H}{\partial p_r}}{\frac{\partial H}{\partial p_\vartheta}} = -\frac{\partial p_\vartheta}{\partial p_r}, \quad \frac{dp_r}{d\vartheta} = p_r' = \frac{\dot{p}_r}{\dot{\vartheta}} = -\frac{\frac{\partial H}{\partial r}}{\frac{\partial H}{\partial p_\vartheta}} = \frac{\partial p_\vartheta}{\partial r}$$

$$\frac{dz}{d\vartheta} = z' = \frac{\dot{z}}{\dot{\vartheta}} = \frac{\frac{\partial H}{\partial p_z}}{\frac{\partial H}{\partial p_\vartheta}} = -\frac{\partial p_\vartheta}{\partial p_z}, \quad \frac{\partial p_z}{\partial \vartheta} = p_z' = \frac{\dot{p}_z}{\dot{\vartheta}} = -\frac{\frac{\partial H}{\partial z}}{\frac{\partial H}{\partial p_\vartheta}} = \frac{\partial p_\vartheta}{\partial z} \quad .$$

Obviously $-p_\vartheta = K(r,\ \vartheta,\ z,\ p_r,\ p_z, H)$, the negative canonical angular momentum acts as the new Hamiltonian, where the function K is found by solving :

$$H(r,\ \vartheta,\ z,\ p_r,\ p_\vartheta,\ p_z) = H = mc^2$$

for p_ϑ :

$$K = -p_\vartheta = -r\left\{\left[(m^2-m_r^2)\ c^2 - (p_r - \frac{e}{c}A_r)^2 - (p_z - \frac{e}{c}A_z)^2\right]^{\frac{1}{2}} + \frac{e}{c}A_\vartheta\right\}$$

$$= -r\left(\left[p^2-(p_r - \frac{e}{c}A_r)^2 - (p_z - \frac{e}{c}A_z)^2\right]^{\frac{1}{2}} + \frac{e}{c}A_\vartheta\right). \qquad (1.1)$$

In the second expression *absolute value of the momentum* $(m^2-m_r^2)c^2 = mv = p$ of the particle has been introduced.

Next we assume a magnetic <u>field</u> which is <u>of rotational symmetry</u> and has no ϑ-component. Then $A_r = A_z = 0$, and the only non-zero component of the vector potential is $A_\vartheta = A(r,z)$.

In an A.G. synchrotron this is true only sector-wise. As to the field in the transition regions from one sector to the next, its influence is small in practice and will be taken account of approximately below. Under these assumptions the Hamiltonian becomes

$$K = -r\left([p^2 - p_r^2 - p_z^2]^{\frac{1}{2}} + \frac{e}{c}A\right) \qquad (I.2)$$

from which the desired Hamiltonian equations follow :

$$\begin{cases} \dfrac{dr}{d\vartheta} = \dfrac{\partial K}{\partial p_r} = \dfrac{r}{[p^2-p_r^2-p_z^2]^{\frac{1}{2}}}\ p_r; \quad \dfrac{dz}{d\vartheta} = \dfrac{\partial K}{\partial p_z} = \dfrac{r}{[p^2-p_r^2-p_z^2]^{\frac{1}{2}}}\ p_z \\[2ex] \dfrac{dp_r}{d\vartheta} = -\dfrac{\partial K}{\partial r} = [p^2-p_r^2-p_z^2]^{\frac{1}{2}} + \dfrac{e}{c}\dfrac{\partial rA}{\partial r} \\[2ex] \dfrac{dp_z}{d\vartheta} = -\dfrac{\partial K}{\partial z} = \dfrac{e}{c}\dfrac{\partial rA}{\partial z} \end{cases} \qquad (I.3)$$

By elimination of p_r, p_z from these equations, second order differential equations for $r(\vartheta)$, and $z(\vartheta)$ are obtained which may be noted in passing :

$$r'' - \frac{r'}{2}\frac{d}{d\vartheta}\log\left[r^2 + (r')^2 + (z')^2\right] - r = \frac{\left[r^2+(r')^2+(z')^2\right]^{\frac{1}{2}}}{p}\frac{\partial}{\partial r}\left(\frac{e}{c}\,rA\right)$$

$$z'' - \frac{z'}{2}\frac{d}{d\vartheta}\log\left[r^2 + (r')^2 + (z')^2\right] = \frac{\left[r^2+(r')^2+(z')^2\right]^{\frac{1}{2}}}{p}\frac{\partial}{\partial z}\left(\frac{e}{c}\,rA\right)$$

These equations contain no approximations, if the field is rotationally symmetric.

Suppose now that $z = o$ is a plane of symmetry of the field, and that a circular "equilibrium" orbit of radius $r = r_o$ exists. From the equations of motion (I.3) follow the conditions for the equilibrium orbit :

$$p_r = 0 \quad , \quad p_z = 0$$

$$p_o + \left(\frac{\partial}{\partial r}\frac{e}{c}\,rA\right)_{\substack{r=r_o\\ z=o}} = p_o + \frac{e}{c}\,r_o\,B_z(r_o,\,o) = p_o + \frac{e}{c}\,r_o B_o = 0$$

$$\left(\frac{\partial}{\partial z}\frac{e}{c}\,rA\right)_{\substack{r=r_o\\ z=o}} = -\frac{e}{c}\,r_o\,B_r(r_o,o) = 0.$$

B_z and B_r are the components of the magnetic flux vector, $B_o = B_z(r_o,o)$ is *the magnetic field guiding a particle of momentum* p_o *on the equilibrium orbit* $r = r_o$. The last one of the conditions is automatically satisfied by the symmetry of the field, the second condition relates the momentum of the equilibrium particle to the magnetic field. Introducing, by

$$r = r_o + x,$$

x and z as deviations from the ideal orbit $r = r_o$, we expand the Hamiltonian (I.2) in polynomial form in p_r, p_z, x, z (remembering that in practice $p_r \ll p$, $p_z \ll p$):

$$K = -(r_o+x)\,p\left(1 - \frac{1}{2}\frac{p_r^2 + p_z^2}{p^2} - \frac{1}{8}\frac{(p_r^2+p_z^2)^2}{p^4} - \ldots\right) - \left(\frac{e}{c}\,rA\right)^{(1)} - \left(\frac{e}{c}\,rA\right)^{(2)} - \ldots$$

$$= -r_o p - px - \left(\frac{e}{c}\,rA\right)^{(1)} + \frac{1}{2}\frac{p_r^2+p_z^2}{p/r_o} - \left(\frac{e}{c}\,rA\right)^{(2)} + \frac{1}{2}\frac{x}{r_o}\frac{p_r^2+p_z^2}{p/r_o} - \left(\frac{e}{c}\,rA\right)^{(3)} +$$

$$= K^{(o)} + K^{(1)} + K^{(2)} + K^{(3)} + \ldots$$

where $\left(\frac{e}{c} rA\right)^{(k)}$ is the polynomial term of degree k in the expansion of $\frac{e}{c} rA$, and $K^{(k)}$ the part of K which is of degree k in the variables. The second degree terms $K^{(2)}$ form the Hamiltonian of a harmonic oscillator of two degrees of freedom, $\frac{p}{r_0} \approx m\dot{\vartheta}$ playing the role of mass.

For convenience we introduce x', z' by

$$x' = \frac{\partial K^{(2)}}{\partial p_r} = r_0 \frac{p_r}{p} , \qquad z' = \frac{\partial K^{(2)}}{\partial p_z} = r_0 \frac{p_z}{p}$$

as new variables to replace p_r, p_z (x', z' are not identical with $\frac{dx}{d\vartheta}, \frac{dz}{d\vartheta}$ except for the purely quadratic Hamiltonian $K^{(2)}$), and a new Hamiltonian by :

$$H = \frac{r_0}{p} K = - r \sqrt{r_0^2 - (x')^2 - (z')^2} - \frac{e}{c} \frac{r_0}{p} rA$$

$$= - r \sqrt{r_0^2 - (x')^2 - (z')^2} + \frac{p_0}{p} \frac{rA}{B_0} .$$

Upon expanding the square root, one finally obtains

$$H = - r_0^2 - r_0 x + V^{(1)} + \frac{(x')^2 + (z')^2}{2} + V^{(2)}$$
$$+ \frac{x}{r_0} \frac{(x')^2 + (z')^2}{2} + V^{(3)} + \frac{1}{r_0^2} \frac{[(x')^2 + (z')^2]^2}{8} + V^{(4)} + \frac{x}{r_0} \frac{1}{r_0^2} \frac{[(x')^2 + (z')^2]^2}{8} + V^{(5)}$$
$$+ \ldots \qquad (I.4)$$

where

$$V^{(1)} + V^{(2)} + V^{(3)} + \ldots = V(x,z) = \frac{p_0}{p} \frac{rA(x,z)}{B_0} \qquad (I.5)$$

is the polynomial expansion of the "potential energy" $V = \frac{p_0}{p} \frac{rA}{B_0}$.

Note that the ideal orbit $r = r_0$ need not be the equilibrium orbit of the actual system. It may be that of some idealized system whose Hamiltonian may be regarded as forming part of the actual Hamiltonian (I.4).

In the magnet gap, the vector potential $\vec{A}$ satisfies : curl curl $\vec{A}=0$ div $\vec{A} = 0$ which leads to the equations for the components :

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial r A_r}{\partial r} + \frac{\partial^2 A_r}{r^2 \partial \vartheta^2} + \frac{\partial^2 A_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial A_\vartheta}{\partial \vartheta} = 0$$

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial r A_\vartheta}{\partial r} + \frac{\partial^2 A_\vartheta}{r^2 \partial \vartheta^2} + \frac{\partial^2 A_\vartheta}{\partial z^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \vartheta} = 0 \qquad (I.6)$$

$$\frac{\partial^2 A_z}{\partial r^2} + \frac{\partial^2 A_z}{r^2 \partial \vartheta^2} + \frac{\partial^2 A_z}{\partial z^2} = 0.$$

Inside a magnet sector the field was supposed to be invariant under rotation, i.e. $A_r = A_z = 0$, and $A_\vartheta = A$ independent of ϑ. The only equation remaining is then

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial rA}{\partial r} + \frac{\partial^2 A}{\partial z^2} = \frac{1}{r}\left(\frac{\partial^2 rA}{\partial r^2} - \frac{1}{r}\frac{\partial rA}{\partial r} + \frac{\partial^2 rA}{\partial z^2}\right) = 0.$$

The potential $V = \frac{p_0}{p}\frac{rA}{B_0}$ appearing in the Hamiltonian (I.4) satisfies therefore

$$\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{I.7}$$

V is needed only in a region whose extension is small in comparison with the radius r_0 of the equilibrium orbit. Within such a region about the equilibrium orbit, V is very near to a solution of Laplace's equation in two dimensions. We, therefore, put

$$V(x,z) = F(x,z) + f(x,z), \qquad x = r - r_0,$$

where F is a Laplace potential obeying

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z^2} = 0, \tag{I.8}$$

and f is a small correction. Substituting in (I.7) we find

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{r_0 + x}\left(\frac{\partial F}{\partial x} + \frac{\partial f}{\partial x}\right) \approx \frac{1}{r_0}\frac{\partial F}{\partial x}$$

neglecting small terms on the r.h.s. We try to satisfy this equation by

$$f = g(x)\,F$$

and obtain the condition for the distortion factor $g(x)$

$$\frac{d^2 g}{dx^2}F + \left(2\frac{dg}{dx} - \frac{1}{r_0}\right)\frac{\partial F}{\partial x} = 0.$$

This equation is satisfied for any F by

$$g = \frac{x}{2r_0}.$$

So

$$V = \left(1 + \frac{x}{2r_0}\right)F(x,z) \tag{I.9}$$

satisfies the right equation (I.7) up to first order terms in $\frac{x}{r_0}$, if $F(x,z)$ is a two-dimensional Laplace potential.

For $F(x,z)$ a polynomial expansion is most convenient, which we may obtain from the general solution of the two-dimensional Laplace equation (I.8) in polar coordinates ρ,φ in the x, z plane :

$$x = \rho \cos \varphi, \quad z = \rho \sin \varphi$$

$$F = \sum_{k=1}^{\infty} F_k \rho^k \cos k (\varphi - \epsilon_k) =$$

$$= \sum_{k=1}^{\infty} F_k \sum_{j=o}^{k} \binom{k}{j} x^{k-j} z^j \cos (k\epsilon_k - j \tfrac{\pi}{2}).$$

This expansion represents a <u>decomposition</u> of F <u>into "multipole" fields</u>, 2k being the order of the multipoles. If $z = o$ is required to be a plane of symmetry, such that $\frac{\partial F}{\partial z} = 0$ in $z = 0$, all ϵ_k have to be zero, and the multipole expansion becomes explicitly :

$$F = F_1 x + F_2(x^2-z^2) + F_3(x^3-3xz^2) + F_4(x^4-6x^2z^2+z^4) + \dots \qquad (I.10)$$

The ϵ_k are the angles by which the planes of symmetry of the component multipoles are twisted. If they are due to construction errors in a synchrotron they can be assumed to be small, and the multipole expansion becomes approximately :

$$F = F_1 x + F_2(x^2-z^2) + F_3 (x^3-3xz^2) + F_4(x^4-6x^2z^2 + z^4) + \dots$$
$$+ \epsilon_1 F_1 z + 2\epsilon_2 F_2 2xz + 3\epsilon_3 F_3(3x^2z - z^3) + 4\epsilon_4 F_4(4x^3z - 4xz^3) + \dots \qquad (I.11)$$

The first line in this expansion is symmetric with respect to the plane $z = 0$, the second line (produced by twist errors) is antisymmetric. The coefficients F_1, F_2, F_3, ... of the symmetric part are completely determined by the z-component $B_z(x,0)$ of the field in the median plane $z = 0$. Similarly, the r-component $B_x(x,0)$ in $z = 0$ determines the antisymmetric part of the field, that is the twist angles ϵ_1, ϵ_2, ... The relations between $B_z(x,0)$ and F_1, F_2,.. *follow from*

$$\left(\frac{\partial V}{\partial x}\right)_{z=o} = \frac{p_o}{p} \frac{1}{B_o} \left(\frac{\partial rA}{\partial x}\right)_{z=o} = \frac{p_o}{p} \frac{B_z(x,o)}{B_o} r = \frac{p_o}{p} \frac{B_z(x,o)}{B_o} r_o \left(1 + \frac{x}{r_o}\right)$$

or

$$\frac{p_o}{p}\,\frac{B_z(x,o)}{B_o}\,r_o = \frac{1}{1+\frac{x}{r_o}}\left\{\left(1+\frac{x}{2r_o}\right)\frac{\partial F}{\partial x} + \frac{1}{2r_o}\,F\right\}$$

$$= F_1 + 2F_2x + 3F_3x^2 + \ldots - \frac{x}{2r_o}(F_2x + 2F_3x^2 + \ldots). \qquad (I.12)$$

Here the curvature correction terms have been kept to first order in $\frac{x}{r_o}$, as (I.9) is correct to this approximation only. For a design aiming at a linear machine the linear term $2F_2x$ in (I.12) will be much greater than the non-linear terms. Then curvature correction terms are negligible because of $\frac{x}{r_o} \ll 1$, except *perhaps the contribution* $\frac{F_2x^2}{2r_o}$ to the quadratic part.

Introducing the field index

$$n(x) = -\frac{r_o}{B_o}\,\frac{\partial B_z(x,0)}{\partial x}, \qquad (I.13)$$

commonly used to characterize the focussing properties of the guiding field, and expanding the l.h.s. of (I.12) into a power series, the coefficients F_1, F_2, can be expressed in terms of $B_z(0,0)$, $n(0)$, and the radial derivatives of n :

$$\frac{p_o}{p}\,\frac{B_z(x,0)}{B_o}\,r_o = \frac{p_o}{p}\left\{\frac{B_z(0,0)}{B_o}\,r_o - n(0)x - \frac{1}{2!}\left(\frac{dn}{dx}\right)_0 x^2 - \frac{1}{3!}\left(\frac{d^2n}{dx^2}\right)_0 x^3 - \ldots\right\}$$

$$= F_1 + 2F_2x + \left(3F_3 - \frac{F_2}{2r_o}\right)x^2 + 4F_4x^3 + \ldots$$

i.e.

$$\begin{cases} F_1 = \frac{p_o}{p}\,\frac{B_z(0,0)}{B_o}\,r_o \\ F_2 = -\frac{p_o}{p}\,\frac{1}{2}\,n(0) \\ F_3 = -\frac{p_o}{p}\,\frac{1}{3!}\left(\frac{dn}{dx}\right)_0 + \frac{p_o}{p}\,\frac{n(0)}{12r_o} \\ F_4 = -\frac{p_o}{p}\,\frac{1}{4!}\left(\frac{d^2n}{dx^2}\right)_0 \\ \text{etc.} \end{cases} \qquad (I.14)$$

Examination of the relative variation $\frac{n(x) - n(0)}{n(0)}$ shows that the contribution of the curvature term left in F_3 is $-\frac{x}{2r_o}$, which is smaller than 1 o/oo all over the cross-section of the vacuum chamber in a machine of the proportions of the CERN Proton Synchrotron. As variations of this magnitude are at the limit of the possibilities of measurement, the non-linear curvature terms shall be neglected altogether in the following. Also for another reason there is no point in taking into account curvature corrections : in practice the magnets of synchrotrons are usually composed of straight magnets blocks arranged in polygon fashion. The disturbance of rotational symmetry caused by this might easily be of the same order of magnitude as the deviations from the two-dimensional field caused by curvature. In fact, with straight blocks the field may even be more nearly two-dimensional.

Completing these results we may therefore adopt as potential

$$V = \left(1 + \frac{x}{2r_o}\right)\left(F_1 x + F_2(x^2 - z^2) + \ldots\right)$$

$$= \frac{p_o}{p}\left[-\frac{B_z(0,0)}{B_o} r_o x + \frac{B_z(0,0)}{B_o}\frac{1}{2}x^2 - \frac{1}{2}n(0)(x^2 - z^2) \right. \tag{I.15}$$

$$\left. - \frac{1}{3!}\left(\frac{dn}{dx}\right)_o (x^3 - 3xz^2) - \frac{1}{4!}\left(\frac{d^2n}{dx^2}\right)_o (x^4 - 6x^2z^2 + z^4) - \ldots \right]$$

Here one term due to curvature and influencing the linear part of the force, namely $\frac{p_o}{p}\frac{B_z}{B_o}\frac{1}{2}x^2$, has been retained. Physically it represents the excess of magnetic bending force over centrifugal force, reponsible for the 1 in the simplified equation

$$\frac{d^2x}{d\vartheta^2} + (1 - n)x = 0$$

for radial betatron oscillations. Though its effect is usually small in an A.G. synchrotron, it is not quite negligible, particularly as it does not alternate like n.

The equations of motion have been set up in cylindrical coordinates, with reference to a circular equilibrium orbit which requires a constant bending field along it. In practice most synchrotrons contain sections without bending field. These *"straight" sections* can be *included in the Hamiltonian* by introducing a new angular position variable η defined by

$$d\eta = \frac{ds_o}{r_m} \quad , \tag{I.16}$$

where ds_0 is the line element along the equilibrium orbit (curved or straight), and the mean radius r_m is defined by

$$\begin{aligned} 2\pi r_m &= \text{total length of equilibrium orbit} \\ &= 2\pi r_0 + \text{total length of straight sections.} \end{aligned} \tag{I.17}$$

Considering first the magnet sectors, it is easily verified that the Hamiltonian $H(x, x', z, z')$ has to be replaced by

$$H^*(x, X', z, Z') = \left(\frac{r_m}{r_o}\right)^2 H\left(x, \frac{r_o}{r_m} X', z, \frac{r_o}{r_m} Z'\right)$$

in order to render the equations of motion with η as independent variable,

$$X' = \frac{r_m}{r_o} x', \quad Z = \frac{r_m}{r_o} z'$$

being the new canonical momenta. Applying these modifications to (I.4), the Hamiltonian inside the curved sectors becomes

$$H^* = -r_m^2 + \frac{1}{2}[(X')^2 + (Z')^2] + \frac{x}{2r_o}[(X')^2 + (Z')^2] + \frac{1}{8r_m^2}[(X')^2 + (Z')^2]^2 + ..$$
$$+ \left(\frac{r_m}{r_o}\right)^2 \left(-r_o x + V\right) .$$

In a straight section the Hamiltonian can be obtained from this as limit for infinite radius of curvature. We replace r_o by r_1 and let $r_1 \to \infty$. In doing so one must be careful with the potential energy term

$$\frac{r_m^2}{r_1^2} V = \frac{r_m^2}{r_1^2} \frac{p_o}{p} \frac{(r_1 + x)A(x,z)}{B_1} = r_m^2 \frac{p_o}{p} \frac{1 + \frac{x}{r_1}}{r_1 B_1} A(x,z)$$

in which

$$\frac{p_o}{r_1 B_1} = \frac{p_o}{r_o B_o} = -\frac{e}{c}$$

remains constant, so that in the limit

$$\frac{r_m^2}{r_1^2} V \to \frac{r_m^2}{r_o^2} \frac{p_o}{p} \frac{r_o A(x,z)}{B_o} ,$$

where r_o, B_o now refer to the curved sectors. The vector potential A in a field free straight section is of course independent of x and z, but the presence of devices like lenses must in general be allowed for.

The Hamiltonian holding in curved as well as in straight sections can now be formulated

$$H^* = \frac{1}{2}\,[(X')^2 + (Z')^2] + \frac{x}{2r_1}\,[(X')^2 + (Z')^2] + \dots + (\frac{r_m}{r_0})^2\,[\,-\frac{r_\vartheta^2}{r_1}\,x + V(x,z)], \tag{I.18}$$

where V and r_1 change discontinuously with η at sector boundaries, and in particular $r_1 = r_0$ in magnet sectors, $r_1 = \infty$ in straight sections.

A discontinuous change of field at sector boundaries is rigorously not possible. Across a boundary between sectors of different field there is a transition region where $A \equiv A_\vartheta$ changes continuously with ϑ and the components A_r, A_z, must appear at the same time. In this region we have to go back to the general Hamiltonian (I.1). Using the Hamiltonian equations which define the canonical momenta,

$$\frac{dr}{d\vartheta} = \frac{\partial K}{\partial p_r} = r\,\frac{p_r - \frac{e}{c}A_r}{[p^2 - (p_r - \frac{e}{c}A_r)^2 - (p_z - \frac{e}{c}A_z)^2]^{\frac{1}{2}}}$$

$$\frac{dz}{d\vartheta} = \frac{\partial K}{\partial p_z} = r\,\frac{p_z - \frac{e}{c}A_z}{[p^2 - (p_r - \frac{e}{c}A_r)^2 - (p_z - \frac{e}{c}A_z)^2]^{\frac{1}{2}}}$$

the equations of motion become

$$\frac{dp_r}{d\vartheta} = [p^2 - (p_r - \frac{e}{c}A_r)^2 - (p_z - \frac{e}{c}A_z)^2]^{\frac{1}{2}} + \frac{e}{c}\frac{\partial}{\partial r}\left(\frac{dr}{d\vartheta}A_r + \frac{dz}{d\vartheta}A_z + rA_\vartheta\right)$$

$$\frac{dp_z}{d\vartheta} = \frac{e}{c}\frac{\partial}{\partial z}\left(\frac{dr}{d\vartheta}A_r + \frac{dz}{d\vartheta}A_z + rA_\vartheta\right) \tag{I.19}$$

If the variations of the field occur within relatively short regions of "<u>fringing field</u>", their effect may be replaced by impuls functions centered on the sector boundaries, and giving the right change of momentum to a particle traversing the fringing field :

$$p_r(\vartheta_2) - p_r(\vartheta_1) = \int_{\vartheta_1}^{\vartheta_2} \frac{dp_r}{d\vartheta}\,d\vartheta =$$

$$\int_{\vartheta_1}^{\vartheta_2} [p^2 - (p_r - \frac{e}{c}A_r)^2 - (p_z - \frac{e}{c}A_z)^2]^{\frac{1}{2}}\,d\vartheta + \frac{e}{c}\int_{\vartheta_1}^{\vartheta_2} \frac{\partial}{\partial r}\left(\frac{dr}{d\vartheta}A_r + \frac{dz}{d\vartheta}A_z + rA_\vartheta\right)d\vartheta$$

$$\approx \frac{e}{c}\frac{\partial}{\partial r}\int_{\vartheta_1}^{\vartheta_2} \vec{A}\, d\vec{s}$$

where ϑ_1, ϑ_2 are situated well inside sectors with longitudinally invariant field, $d\vec{s}$ has components dr, $rd\vartheta$, dz, and r, $\frac{dr}{d\vartheta}$, $\frac{dz}{d\vartheta}$ have been assumed to be sufficiently constant over the fringing field region. The latter is justified by the small angles between equilibrium orbit and particle trajectories. Similarly

$$p_z(\vartheta_2) - p_z(\vartheta_1) \approx \frac{e}{c}\ \frac{\partial}{\partial z}\int_{\vartheta_1}^{\vartheta_2} \vec{A}\, d\vec{s}$$

The quantity $\int_{\vartheta_1}^{\vartheta_2} \vec{A}\, d\vec{s}$ is related to the magnetic flux through a surface σ bounded by the particle trajectory $\vec{s}(\vartheta)$, the ideal orbit trajectory $\vec{s}_0$ and ϑ_1 and ϑ_2 :

$$\int_{\vartheta_1}^{\vartheta_2}\left(\vec{A}\ (s) - \vec{A}(s_0)\right) d\vec{s} = \oint \vec{A}\ d\vec{s} = \int \text{curl}\ \vec{A}\ d\vec{s} = \int \vec{B}\ d\vec{\sigma}\ .$$

As $\frac{dr}{ds}$ and $\frac{dz}{ds}$ are very small the effect of the inclination of the orbit on $\int \vec{B}\ d\vec{\sigma}$ can be neglected, and $\int_{\vartheta_1}^{\vartheta_2} \vec{A}\ d\vec{s}$ becomes a function of r, z only. Writing the field as a sum of an idealized field $\vec{A}^{disc}$ changing discontinuously at the sector boundary plus the fringing field :

$$\vec{A} = \vec{A}^{disc} + (\vec{A} - \vec{A}^{disc})\ ,$$

the equations of motion can be written in the form

$$\frac{dp_r}{d\vartheta} = [p^2 - p_r^2 - p_z^2]^{\frac{1}{2}} + \frac{e}{c}\frac{\partial}{\partial r}\left(rA_\vartheta^{disc} + \phi(r,z)\ \delta\ (\vartheta - \vartheta_b)\right)$$

$$\frac{dp_z}{d\vartheta} = \frac{e}{c}\frac{\partial}{\partial z}\left(rA_\vartheta^{disc} + \phi(r,z)\ \delta\ (\vartheta - \vartheta_b)\right)$$

where $$\phi(r,z) = \int_{\vartheta_b-\epsilon}^{\vartheta_b+\epsilon}\left(\vec{A}(r,z,\vartheta) - \vec{A}^{disc}(r,z,\vartheta)\right) d\vec{s}$$

and ϑ_b the position of the sector boundary under consideration. The *fringing* field can therefore be described approximately by a δ-function potential to be added to rA_ϑ^{disc} . As to the dependence on r,z, $\phi(r,z)$ obeys the same modified Laplace equation (I.7) as rA_ϑ^{disc} obeys inside uniform sectors :

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\int_{\vartheta_1}^{\vartheta_2} \vec{A}\, d\vec{s} \approx$$

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\int_{\vartheta_1}^{\vartheta_2} rA_\vartheta d\vartheta = -\int_{\vartheta_1}^{\vartheta_2} \frac{\partial}{\partial\vartheta}\left(\frac{2A_r}{r} + \frac{1}{r}\frac{\partial A_\vartheta}{\partial\vartheta}\right) d\vartheta = 0,$$

as $A_r = \frac{\partial A_\vartheta}{\partial\vartheta} = 0$ at ϑ_1 and ϑ_2. Here one of the formulae (I.6) expressing curl curl $\vec{A} = 0$ has been used.

Redefining a fringing field potential V^{fr} in accordance with the definition of the potential V in (I.5), the radial force due to the fringing field can be written

$$-\frac{\partial V^{fr}}{\partial x} = \frac{e}{c}\frac{r_0}{p}\frac{\partial\phi(x,z)}{\partial x}\,\delta(\vartheta - \vartheta_b) = -\frac{p_0}{pB_0}\frac{\partial\phi}{\partial x}\,\delta(\vartheta - \vartheta_b)$$

$$= -\frac{p_0}{pB_0}\int_{\vartheta_b-\epsilon}^{\vartheta_b+\epsilon} (B_z - B_z^{disc})\, ds\, \delta(\vartheta - \vartheta_b)$$

(This formula is, by the way, immediately evident). It is convenient to express the fringing field in terms of an equivalent change of length of the discontinuous field sectors which would produce the same force. Thus, to produce the effect of the fringing field between a magnet sector and a field free section, the effective length of a discontinuous magnet sector must be increased by

$$\Delta\ell(x,z) = \frac{\int_{\vartheta_b-\epsilon}^{\vartheta_b+\epsilon}\left(B_z(x,z,\vartheta) - B_z^{disc.}(x,z,\vartheta)\right) ds}{B_z^{disc}(x,z)}$$

where $B_z^{disc}(x,z)$ is the discontinuous field inside the magnet sector. In the case of a boundary between two adjacent magnet sectors (e.g. a + |n| and a - |n| sector), the effective lengths of each sector are changed by

$$\Delta\ell^-(x,z) = \frac{\int_{\vartheta_b-\epsilon}^{\vartheta_b}(B_z - B_z^{disc})\, ds}{B_z^{disc\,-}} \quad , \quad \Delta\ell^+ = \frac{\int_{\vartheta_b}^{\vartheta_b+\epsilon}(B_z - B_z^{disc})\, ds}{B_z^{disc\,+}} ,$$

$\mp$ indicating the sectors preceding and following $\vartheta = \vartheta_b$.

Effective length is a useful concept because it can be measured relatively easily.

Once $\Delta\ell(x,0)$ is known in the median plane, V^{fr} can be found on account of its general form

$$V^{fr} = V_1^{fr}\, x + V_2^{fr}\,(x^2 - z^2) + V_3^{fr}\,(x^3 - 3xz^2) + \ldots$$

by comparing

$$\begin{aligned} -\frac{\partial V^{fr}(x,0)}{\partial x} &= -V_1^{fr} - 2V_2^{fr}\,x - 3V_3^{fr}\,x^2 \\ &= -\frac{p_o}{p}\,\frac{B_z^{disc}(x,0)}{B_o}\,\Delta\ell(x,0)\;\delta(\vartheta - \vartheta_b) \qquad \text{(I.20)} \\ &= -\left[1 - n^{disc}(0,0)\,\frac{x}{r_o}\right]\Delta\ell(x,0)\;\delta(\vartheta - \vartheta_b) \end{aligned}$$

Here $\frac{p_o}{p} = 1$ and $\frac{B_z^{disc}(0,0)}{B_o} = 1$ have been assumed, and non-linear terms of $B_z^{disc}(x,0)$ omitted, as we are calculating an effect which is itself small.

To illustrate orders of magnitude, figures found for the end face of a CERN PS magnet (from measurements on a full scale model) may be given. These magnets are designed to produce a relative field gradient

$$\frac{1}{B_o}\,\frac{\partial B_z}{\partial x} = 0.040\ \text{cm}^{-1}$$

in a gap of height 10 cm at the ideal orbit. The effective length of the fringing field (one end face) is

$$\Delta\ell = \Delta\ell_o + \left(\frac{d\Delta\ell}{dx}\right)_o x = 7 \pm 0.15\,x\ \text{cm, for } n \gtrless 0. \qquad \text{(I.21)}$$

It increases roughly linearly towards the "open" side of the gap. Using (I.20 + 21), we may write finally

$$V^{fr} = \left\{\Delta\ell_o\, x - \frac{1}{2}\left[n\,\frac{\Delta\ell_o}{r_o} - \left(\frac{d\Delta\ell}{dr}\right)_o\right](x^2 - z^2) - \frac{1}{3}\,\frac{n}{r_o}\left(\frac{d\Delta\ell}{dx}\right)_o (x^3 - 3xz^2)\right\}\delta(\vartheta - \vartheta_b)$$

(I.22)

This has to be multiplied by $(\frac{r_m}{r_o})^2$ if $\eta = \frac{r_o}{r_m}\,\vartheta$ is used as azimuthal variable. At the same time $\delta(\vartheta - \vartheta_b)$ must be replaced by : $\frac{r_0}{r_m}\,\delta(\eta - \eta_b)$.

The Hamiltonian has to be extended still further to include accidental imperfections. They can be due to (a) misalignment of individually perfect sectors, (b) non-uniformity of the field from sector to sector, or even within the single sectors.

Rigorously, the non-uniformity of the field implies the appearance of A_r, A_z components of the vector potential as in the case of fringing fields. Going back again to the general Hamiltonian and the resulting equations of motion (I.19), the contribution to the forces due to the small accidental components A_r, A_z is seen to be small in comparison with the contribution of the accidental disturbance of A_ϑ, $\frac{dr}{d\vartheta}$ and $\frac{dz}{d\vartheta}$ being small quantities. In physical terms, this statement means that the transversal force due to longitudinal particle velocity $v_\parallel$ and transversal field disturbances δB_x, δB_z is much more important than that due to transversal velocity $v_\perp$ and longitudinal field disturbance δB_ϑ. It is therefore justified to take solely the disturbance of A_ϑ into consideration in calculating the effects of imperfections.

In order to express the perturbation of A_ϑ in terms of misalignments of magnets, we fix in a magnet unit a coordinate frame $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ coinciding with $x \equiv x_1$, $r\vartheta \equiv x_2$, $z \equiv x_3$ if the unit is in the correct position. A unit can be considered as straight for the present purpose. In the unit frame we have the unperturbed field given by

$$V = \frac{p_0}{p}\,\frac{rA_\vartheta}{B_0} = F_1\bar{x}_1 + F_2(\bar{x}_1^2 - \bar{x}_3^2) + \ldots$$

In the machine frame x_1, x_2, x_3 the field is found by applying the coordinate transformation between the two frames :

$$\bar{x}_i = \sum_{j=1}^{3} \alpha_{ij}(x_j - \xi_j)$$

where ξ_1, ξ_2, ξ_3 are the displacements of the center of the unit, and the matrix α_{ij} can be written

$$\alpha_{ij} = \begin{bmatrix} 1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 1 & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & 1 \end{bmatrix} = \delta_{ij} + \epsilon_{ij}$$

for infinitesimal angles ϵ_{ij} of rotation of the unit. e.g. ϵ_{12} is the angle by which the unit must be rotated about the 3-axis in the sense $1 \to 2$ to bring the $\bar{1}$ axis of the unit in line with the 1,3-plane of the machine frame. Thus ϵ_{12}, ϵ_{23} are tilts about the vertical and radial axis, ϵ_{13} is a twist about the orbit axis. As α_{ij} is an orthogonal matrix, ϵ_{ij} is anti-symmetric

$$\epsilon_{21} = -\epsilon_{12} , \quad \epsilon_{23} = -\epsilon_{32} , \quad \epsilon_{31} = -\epsilon_{13} .$$

A_1, A_2, A_3 transform like x_1, x_2, x_3, so we have in the machine frame

$$A_\vartheta = A_2 = \sum \bar{\alpha}^{-1}_{2k} \bar{A}_k = \bar{\alpha}^{-1}_{22} \bar{A}_2$$

$$\bar{\alpha}^{1}_{jk} = \delta_{jk} + \epsilon_{kj} = \delta_{jk} - \epsilon_{jk}$$

being the inverse matrix to α_{ij}, and

$$V = \bar{\alpha}^{1}_{22} (F_1\bar{x}_1 + F_2(\bar{x}_1^2 - \bar{x}_3^2) + ..)$$

$$= F_1[(x_1-\xi_1) + \epsilon_{12}(x_2-\xi_2) + \epsilon_{13}(x_3-\xi_3)]$$

$$F_2 \left[\left((x_1-\xi_1) + \epsilon_{12}(x_2-\xi_2) + \epsilon_{13}(x_3-\xi_3)\right)^2 - \left((x_3-\xi_3) + \epsilon_{31}(x_1-\xi_1)+\epsilon_{32}(x_2-\xi_2)\right)^2 \right]$$

$$\approx F_1[x_1 + \epsilon_{13}x_3] + F_2[x_1^2-x_3^2-2\xi_1 x_1+2\xi_3 x_3+2\epsilon_{12}x_1 x_2+2\epsilon_{23}x_2 x_3+4\epsilon_{13}x_1 x_3]$$

to first order terms in the ξ's and ϵ's. Terms not dependent on x_1 or x_3 can be omitted from the potential.

$$\xi_1 - \epsilon_{12}x_2 = \xi(\vartheta)$$

$$\xi_3 + \epsilon_{23}x_2 = \zeta(\vartheta)$$

combine to form the local displacements $\xi(\vartheta)$ and $\zeta(\vartheta)$ of the unit orbit axis in radial and vertical direction. Rewritten in the old notation, after having used (I.14) to express F_1 and F_2 by B_z and n, the perturbed potential reads

$$V = \frac{p_0}{p} \left\{ \frac{B_z}{B_0} r_0 x - \frac{1}{2} n (x^2-z^2) - \dots \right.$$
$$+ \epsilon \frac{B_z}{B_0} r_0 z - 2\epsilon n(\vartheta) xz$$
$$+ n(\vartheta)\, \xi(\vartheta) x - n(\vartheta)\zeta(\vartheta) z$$
$$\left. + \frac{B_z(\vartheta+\delta\vartheta) - B_z(\vartheta)}{B_0} r_0 x - \frac{1}{2} [n(\vartheta+\delta\vartheta) - n(\vartheta)](x^2-z^2) \right\} \qquad (I.23)$$

Here $\epsilon = \epsilon_{13}$ is the angle by which the unit is twisted about the orbit axis (in the sense $x \to z$). The terms arising from the twist produce a vertical force distorting the equilibrium orbit and a coupling force, and are exactly those to be expected from the general multipole expansion (I.11) as a consequence of removing the plane of symmetry. The terms arising from transverse *displacements together with the field gradient, produce horizontal and vertical* forces affecting the equilibrium orbit. Lastly, a longitudinal displacement $\delta\vartheta$ produces perturbations of the guiding and focussing forces corresponding to the difference between actual and ideal field appearing at sector ends.

Now at last a Hamiltonian for the A.G. synchrotron can be formulated, comprising all of the more important features. We have to use the modified angular position variable η, whose element is $\frac{ds_0}{r_m}$; we will, however, write ϑ ($\equiv \eta$) for it, as the former ϑ no longer appears in the following. Consequently *X', Z' will also be written x', z'. Supplementing (I.18) by (I.15)* (I.22), (I.23) we obtain

$$H = H^{(0)} + H^{(1)}$$

$$H^{(0)} = \frac{1}{2} [(x')^2+(z')^2] - \frac{1}{2} \left(\frac{r_m}{r_0}\right)^2 n(x^2-z^2)$$

$$H^{(1)} = \left(\frac{r_m}{r_0}\right)^2 \frac{p_0}{p} \left\{ \left[\frac{\delta B_z}{B_0} r_0 - \frac{r_0}{r_1} \frac{\delta p}{p_0} r_0 + \frac{r_0}{r_m} \Delta\ell + n\xi \right] x + [\epsilon r_0 - n\zeta] z \right.$$
$$+ \frac{1}{2} \frac{r_0}{r_1} x^2 - \frac{1}{2} \left[\delta n - \frac{\delta p}{p_0} n + \frac{r_0}{r_m} \left(\frac{\Delta\ell}{r_0} n - \frac{d\Delta\ell}{dx}\right) \right] (x^2-z^2) - 2\epsilon nxz$$
$$\left. - \frac{1}{3!}\left(\frac{dn}{dx} + \frac{2n}{r_m} \frac{d\Delta\ell}{dx}\right) (x^3-3xz^2) - \frac{1}{4!} \left(\frac{d^2n}{dx^2}\right) (x^4-6x^2z^2 + z^4) - \dots \right\}$$
$$+ \frac{x}{2r_1} [(x')^2+ (z')^2] + \frac{1}{8r_m^2} [(x')^2+ (z')^2]^2 + \dots$$

$$= \quad V_{10}x + V_{01}z$$

$$+ V_{20}x^2 + V_{02}z^2 + V_{11}xz$$

$$+ V_3(x^3 - 3xz^2) + V_4(x^4 - 6x^2z^2 + z^4) + \ldots$$

$$+ \frac{x}{2r_1}[(x')^2 + (z')^2] + \frac{1}{8r_1^2}[(x')^2 + (z')]^2 + \ldots \qquad (I.24)$$

The Hamiltonian has been divided into two parts : $H^{(0)}$ characterizing an idealized linear system, and $H^{(1)}$ containing all deviations and perturbations. $H^{(1)}$ is written in the form of a polynomial expansion in the canonical variables, up to the 4th degree (cubic force terms). Most of the coefficients are functions of ϑ : $r_1 = r_0$ in curved sections and $r_1 = \infty$ in straight sections. δB_z and δn incorporate systematic deviations from the nominal values B_0 and n on the ideal orbit (e.g. due to lenses and length adjustments of sectors) as well as accidental ones (due to *azimuthal misalignments and quality fluctuations*). $\Delta\ell$ *and* $\frac{d\Delta\ell}{dx}$ *stand as* abbreviations for δ-function kicks appearing at sector ends and accounting for fringing fields; the integrated δ-functions represent effective length corrections $\Delta\ell$, and change of effective length with radius $\frac{d\Delta\ell}{dx}$ on the ideal orbit.

The symbols $V_{k_1k_2}$ serve as abbreviations for the coefficients of $H^{(1)}$. When treating one dimensional motions ($z \equiv 0$), one subscript $k = k_1+k_2$ is sufficient.

The *terms in* $H^{(1)}$ *containing momenta* have been omitted in the analysis of the preceding sections to simplify the presentation. For the CERN Proton Synchrotron, the fourth degree term is small enough to be neglected. The third degree term may, however, contribute noticeably to the quadratic non-linearity. There is no difficulty in keeping this term in the perturbation Hamiltonian in the investigation of a practical case.

Appendix II. Theory of the idealized linear "unperturbed" A.G.S.*

The Hamiltonian of the A.G.S. has been divided into a first part $H^{(0)}$ holding for an idealized system plus a second part $H^{(1)}$ containing the systematic and accidental additions characterizing the real physical system (see section 3 and Appendix I). The perturbation theory employed in this report is based on the solutions for the idealized system in which x and z obey the Hill-type differential equations

$$\frac{d^2x}{d\vartheta^2} - \frac{r_m^2}{r_o^2}\, n(\vartheta)x = 0 \qquad \text{(II.1)}$$

$$\frac{d^2z}{d\vartheta^2} + \frac{r_m^2}{r_o^2}\, n(\vartheta)z = 0 \ . \qquad \text{(II.2)}$$

ϑ is the azimuthal position variable defined by (I.16) as length along the equilibrium orbit divided by the mean radius r_m . n is as defined in (I.13) (with reference to the radius r_o of curvature in the magnet sectors).

If the A.G. structure is to provide equally good focussing in x- and z-direction, $n(\vartheta)$ must alternate between equal and opposite values with equal length of $-|n|$ and $+|n|$ sectors (also called F- and D-sectors respectively, because there is radial focussing in negative n and radial defocussing in positive n sectors). The most simple structure of this type is the pure square wave n illustrated by fig.15a. Fig.15b shows a structure with field free (straight) sections in the middle of each F- and D-sector; This type of structure is used in the CERN P.S.

The solution of (II.1) giving $x(\vartheta)$, $x'(\vartheta) = \frac{dx}{d\vartheta}$ in terms of $x(0)$, $x'(0)$ at some initial position $\vartheta = 0$ is most conveniently written by means of the <u>transfer matrix</u> T :

$$\begin{pmatrix} x(\vartheta) \\ x'(\vartheta) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = T \quad \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} \qquad \text{(II.3)}$$

Regarding the solutions of (II.1) the transfer matrix for a section of constant n is easily seen to be

$$\text{for a } -|n| \text{ or F-sector : } \begin{bmatrix} \cos\left(\sqrt{n}\,\frac{r_m}{r_o}\,\vartheta\right) & \dfrac{\sin\left(\sqrt{n}\,\frac{r_m}{r_o}\,\vartheta\right)}{\frac{r_m}{r_o}\sqrt{n}} \\ \frac{-r_m}{r_o}\sqrt{n}\,\sin\left(\sqrt{n}\,\frac{r_m}{r_o}\,\vartheta\right) & \cos\left(\sqrt{n}\,\frac{r_m}{r_o}\,\vartheta\right) \end{bmatrix} \qquad \text{(II.4)}$$

* This appendix is included for convenient reference to otherwise well known results of transfer matrix theory.

for a $+|n|$ or D-sector: $$\begin{bmatrix} \cosh\left(\sqrt{n}\,\frac{r_m}{r_0}\,\vartheta\right) & \dfrac{\sinh\left(\sqrt{n}\,\frac{r_m}{r_0}\,\vartheta\right)}{\frac{r_m}{r_0}\sqrt{n}} \\ \frac{r_m}{r_0}\sqrt{n}\,\sinh\left(\sqrt{n}\,\frac{r_m}{r_0}\,\vartheta\right) & \cosh\left(\sqrt{n}\,\frac{r_m}{r_0}\,\vartheta\right) \end{bmatrix} \qquad (II.5)$$

for $n = 0$ (straight section) $$\begin{pmatrix} 1 & \vartheta \\ 0 & 1 \end{pmatrix} \qquad (II.6)$$

For A.G. structures like those illustrated by fig.15, the transfer matrix is a product composed of elementary matrices of type (II.4-6).

The determinant of the matrices (II.4-6), and thus of all transfer matrices is unity. This is an expression of Liouville's theorem stating the conservation of area in the phase plane x, x' in the course of the transformation by the motion.

Floquet's theorem which states that the solution (II.3) can be written in the form (3.2) follows by introducing the eigenvectors of the transfer matrix $T(\Theta)$ over one complete period $\Theta = \frac{2\pi}{M}$ of $n(\vartheta)$. These are defined by the eigenvalue problem

$$T(\Theta)\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \lambda \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} . \qquad (II.7)$$

The transfer matrix over one period, and therefore the eigenvectors, depend on the starting point of the period. The notation

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$$

indicates that $\vartheta = 0$ has been chosen as a starting point. The eigenvalues are roots of the "characteristic equation"

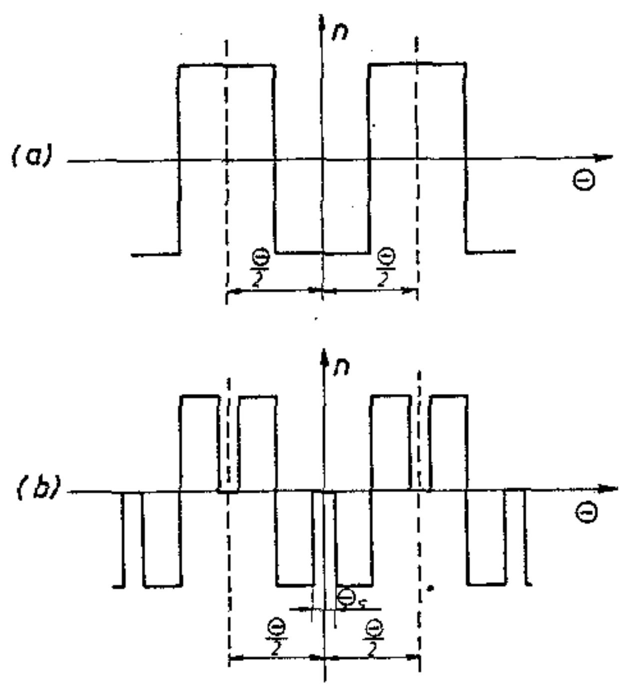

Fig. 15 (a) $n(\vartheta)$ in simple A.G. structure.
(b) $n(\vartheta)$ in A.G. structure with field free sections in the middle of focusing and defocusing sectors.

$$\begin{vmatrix} T_{11} - \lambda & T_{12} \\ T_{21} & T_{22} - \lambda \end{vmatrix} = \lambda^2 - \lambda(T_{11} + T_{22}) + \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix}$$

$$= \lambda^2 - \lambda(T_{11} + T_{22}) + 1 = 0,$$

showing that the two roots λ_1, λ_2 must satisfy the relations

$$\lambda_1 + \lambda_2 = T_{11} + T_{22}$$
$$\lambda_1 \lambda_2 = 1 \quad ,$$

which suggest the notation

$$\lambda_1 = e^{iQ\Theta}$$
$$\lambda_2 = e^{-iQ\Theta}$$

$$2 \cos Q\Theta = T_{11} + T_{22} \tag{II.8}$$

Q will be real or complex, depending on whether $T_{11} + T_{22}$ is smaller or greater than 2.

Q is independent of the choice of the starting point of the period, as can be shown easily.

The eigenvectors are found immediately from (II.7) :

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{\substack{(1)\\(2)}} = \begin{pmatrix} T_{12} \\ e^{\pm iQ\Theta} - T_{11} \end{pmatrix} \tag{II.9}$$

Considering now an interval $\vartheta = N\Theta + \vartheta'$, and representing the initial vector $\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$ as linear combination of the eigenvectors

$$\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = C_1 \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{(1)} + C_2 \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{(2)}$$

the solution (II.3) becomes

$$\begin{pmatrix} x(\vartheta) \\ x'(\vartheta) \end{pmatrix} = T(\vartheta') \, T(\Theta)^N \left[C_1 \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{(1)} + C_2 \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{(2)} \right] \tag{II.10}$$

$$= C_1 \, e^{iQN\Theta} \, T(\vartheta') \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{(1)} + C_2 \, e^{-iQN\Theta} \, T(\vartheta') \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}_{(2)}$$

In the case of complex Q, one of the fundamental parts of this solution increases exponentially with the number N of periods traversed. On the other hand, for real Q both parts remain bounded.

$$Q = \text{real} \quad \text{or} \quad |\cos Q\Theta| = \left|\frac{T_{11} + T_{22}}{2}\right| < 1$$

is therefore the fundamental condition for the A.G. structure to provide focussing.

Assuming this condition to be satisfied, the eigenvectors (9) and the coefficients C_1, C_2 are conjugate complex (because the components of T and $\binom{x}{x'}$ are real) so that (II.10) can be written, after having reintroduced $\vartheta = N\Theta + \vartheta'$,

$$\begin{pmatrix} x(\vartheta) \\ x'(\vartheta) \end{pmatrix} = c e^{iQ\vartheta} e^{-iQ\vartheta'} T(\vartheta') \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} + \overset{*}{c} e^{-iQ\vartheta} e^{iQ\vartheta'} T(\vartheta') \begin{pmatrix} \overset{*}{w}_1(0) \\ \overset{*}{w}_2(0) \end{pmatrix} \tag{II.11}$$

which is the Floquet form of solution used in (3.2), exhibiting the Floquet factors

$$\begin{pmatrix} w_1(\vartheta) \\ w_2(\vartheta) \end{pmatrix} = e^{-iQ\vartheta} T(\vartheta) \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}, \quad 0 < \vartheta < \Theta \tag{II.12}$$

as functions with period Θ :

$$w_1(\vartheta + \Theta) = w_1(\vartheta)$$

$$w_2(\vartheta + \Theta) = w_2(\vartheta) .$$

A few general properties of transfer matrices are useful to facilitate the calculation of the matrices for the structures we are concerned with.

If a section with matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

is reversed, then the matrix of the reversed section is obtained by (i) taking the inverse matrix T^{-1} and (ii) reversing the direction of ϑ. The result is

$$T^{rev} = \begin{pmatrix} T_{22} & T_{12} \\ T_{21} & T_{11} \end{pmatrix} .$$

If the structure of a section is symmetric with respect to its center, then

$$T^{rev} = T, \quad \text{that is} \quad T_{22} = T_{11}$$

must hold.

Adding to any section its reversed one results in a reversible symmetric structure element. The trace of such a combination is therefore

$$2(T^{rev}\,T)_{11} = 2(T_{22}T_{11} + T_{12}T_{21}) \;.$$

Proceeding to the calculation of Q-values for the specified structures of fig. 15, one obtains for (a), the simple square wave structure,

$$T(\theta) = \begin{bmatrix} \cosh\sqrt{n}\,\frac{\theta}{2} & \frac{\sinh\sqrt{n}\,\frac{\theta}{2}}{\sqrt{n}} \\ \sqrt{n}\,\sinh\sqrt{n}\frac{\theta}{2} & \cosh\sqrt{n}\,\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos\sqrt{n}\frac{\theta}{2} & \frac{\sin\sqrt{n}\frac{\theta}{2}}{\sqrt{n}} \\ -\sqrt{n}\,\sin\sqrt{n}\frac{\theta}{2} & \cos\sqrt{n}\,\frac{\theta}{2} \end{bmatrix}$$

having the characteristic equation

$$\cos Q\theta = \frac{T_{11} + T_{22}}{2} = \cosh\sqrt{n}\,\frac{\theta}{2}\,\cos\sqrt{n}\,\frac{\theta}{2} \;. \qquad (\text{II}.13)$$

For the structure (b) with field free sections of length θ_s each the transfer matrix over one full period can be written

$$T(\theta) = S\;F\;D\;S\cdot S\;D\;F\;S = (S\;D\;F\;S)^{rev}\,(S\;D\;F\;S),$$

introducing the symbols S, F, D for the matrices of one half straight, one half F-, and one half D-section :

$$S = \begin{pmatrix} 1 & \frac{\theta s}{2} \\ 0 & 1 \end{pmatrix}, \quad F = \begin{bmatrix} \cos\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right) & \frac{\sin\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right)}{\frac{r_m}{r_o}\sqrt{n}} \\ -\frac{r_m}{r_o}\sqrt{n}\sin\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4}-\frac{\theta s}{2}\right) & \cos\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right) \end{bmatrix}$$

$$D = \begin{bmatrix} \cosh\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right) & \frac{\sinh\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right)}{\frac{r_m}{r_o}\sqrt{n}} \\ \frac{r_m}{r_o}\sqrt{n}\sinh\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right) & \cosh\sqrt{n}\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta s}{2}\right) \end{bmatrix}$$

As $\theta - 2\theta_s = \frac{r_o}{r_m}\theta$, the arguments of the circular and hyperbolic functions can also be written

$$\sqrt{n}\,\frac{r_m}{r_o}\left(\frac{\theta}{4} - \frac{\theta_s}{2}\right) = \sqrt{n}\,\frac{\theta}{4} \quad .$$

For this structure, the characteristic equation results :

$$\begin{aligned}\cos Q\theta = {} & \cos\sqrt{n}\tfrac{\theta}{2}\,\cosh\sqrt{n}\tfrac{\theta}{2} \\ & + \frac{2\theta_s r_m}{r_o}\sqrt{n}\,\cos\sqrt{n}\tfrac{\theta}{4}\,\cosh\sqrt{n}\tfrac{\theta}{4}\left(\cos\sqrt{n}\tfrac{\theta}{4}\,\sinh\sqrt{n}\tfrac{\theta}{4} - \sin\sqrt{n}\tfrac{\theta}{4}\,\cosh\sqrt{n}\tfrac{\theta}{4}\right) \\ & + \frac{1}{2}\left(\frac{\theta_s r_m}{r_o}\right)^2 n\left(\cos\sqrt{n}\tfrac{\theta}{4}\,\sinh\sqrt{n}\tfrac{\theta}{4} - \sin\sqrt{n}\tfrac{\theta}{4}\,\cosh\sqrt{n}\tfrac{\theta}{4}\right)^2 . \end{aligned} \qquad \text{(II.14)}$$

The n-values resulting from (II.13) and (II.14) for structures (a) and (b) and the data of the CERN PS can be found in the table of Appendix IV. They are $n = 336.5$ for (a) and $n = 282.4$ for (b). (It would be wrong to conclude from these two figures that less gradient is necessary in the case (b) with field free sections to obtain the same focussing power (measured by Q). Comparing the relative field gradients

$$\frac{1}{B}\frac{\partial B}{\partial x} = -\frac{n}{r_o},$$

they are $\frac{336}{10000}$ and $\frac{282}{7008}$, that is really larger in the second case as one would expect).

The <u>Floquet factors</u> (II.12) become rather tedious expressions, even for the most simple structure (a), after explicit substitution of the transfer matrix components. Fortunately, it is easy to calculate the Fourier coefficients of the Floquet factors, as will be shown in Appendix IV. These Fourier coefficients are all we need for the evaluation of our perturbation theory.

Furthermore the knowledge of $w_1(\vartheta)$ is sufficient because

$$w_2(\vartheta) = iQ\,w_1(\vartheta) + \frac{dw_1}{d\vartheta} \quad . \qquad \text{(II.15)}$$

This relation is obtained by differentiation of $x(\vartheta)$ as given by the first line of the Floquet form of solution (3.2) or (II.12).

The Floquet factors $\begin{matrix} u_1(\vartheta) \\ u_2(\vartheta) \end{matrix}$ for the z-motion differ from those for the x-motion, because of the opposite signs of $n(\vartheta)$ in the equations

(II.1) and (II.2). However, for structures of the kind we are considering here

$$n(\vartheta) = - n(\vartheta + \frac{\Theta}{2})$$

is obtained by shifting the x-structure half a period. Therefore

$$\begin{pmatrix} z(\vartheta) \\ z'(\vartheta) \end{pmatrix} = \begin{pmatrix} x(\vartheta + \frac{\Theta}{2}) \\ x'(\vartheta + \frac{\Theta}{2}) \end{pmatrix} ,$$

leading to the following relation between the Floquet factors for z and x :

$$u_1(\vartheta) = w_1(\vartheta + \frac{\Theta}{2})\, e^{\frac{iQ\Theta}{2}}$$

$$u_2(\vartheta) = w_2(\vartheta + \frac{\Theta}{2})\, e^{\frac{iQ\Theta}{2}} .$$

The constant factor $e^{\frac{iQ\Theta}{2}}$ can be dropped, so that we can take :

$$\begin{pmatrix} u_1(\vartheta) \\ u_2(\vartheta) \end{pmatrix} = \begin{pmatrix} w_1(\vartheta + \frac{\Theta}{2}) \\ w_2(\vartheta + \frac{\Theta}{2}) \end{pmatrix} \qquad \text{(II.16)}$$

as Floquet vector for z. It is automatically normalized like $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

Appendix III. Fourier analysis of A.G. synchrotron structure parameters

As far as the basic structure, defined by $n(\vartheta)$, and the systematic modifications by sector length corrections, non-linearities and lenses (linear and non-linear) go, we have to deal with periodic repetitions of given structures round the machine.

Repeating a function $g(\vartheta - \vartheta_k)$, centered at

$$\vartheta_k = k\,\frac{2\pi}{P} + \vartheta_o,\ k,\ P \text{ integers,}$$

P times round the orbit at equidistant points $\frac{2\pi}{P}$ apart, results in a function

$$f(\vartheta) = \sum_{k=1}^{P} g(\vartheta-\vartheta_k) = \sum_{\nu P=-\infty}^{+\infty} f_{\nu P}\, e^{-i\nu P\vartheta}$$

with fundamental period $\frac{2\pi}{P}$, and Fourier coefficients

$$f_{\nu P} = g_{\nu P}\; P e^{i\nu P\vartheta_o}\ , \qquad \text{(III.1)}$$

where

$$g_{\nu P} = \frac{1}{2\pi}\int_0^{2\pi} g(\vartheta)e^{i\nu P\vartheta}\, d\vartheta\ .$$

All we need as structure element $g(\vartheta)$ for our present purpose is a rectangular pulse. If it has height g and width η,

$$g(\vartheta) = \begin{cases} 0 \text{ in} & \vartheta < -\frac{\eta}{2} \\ \hat{g} \text{ in} & -\frac{\eta}{2} < \vartheta < \frac{\eta}{2} \\ 0 \text{ in} & \frac{\eta}{2} < \vartheta \end{cases}$$

and

$$g_{\nu P} = \frac{g\eta}{2\pi}\;\frac{\sin\frac{\nu P\eta}{2}}{\frac{\nu P\eta}{2}}\ . \qquad \text{(III.2)}$$

We use these formulae to calculate the Fourier coefficients of the most important structure parameters :

(i) field index $n(\vartheta)$:

(a) basic structure without field free sections, according to fig.15 a : a F-sector (n negative) and a D-sector (n positive), each of length $\frac{\Theta}{2} = \frac{\pi}{M}$ is repeated M times. Starting at $\vartheta_o = 0$ with the center of a sector with n-value $\hat{n}$, the centers of sectors with $-\hat{n}$ start at $\vartheta_0 = \frac{\Theta}{2}$. If the sector at $\vartheta_0 = 0$ is an F-sector, $\hat{n}$ would be negative $=-|\hat{n}|$. The Fourier coefficients of $n(\vartheta)$ follow immediately from (III.2) and (III.1) as

$$n_{\nu M} = \hat{n}\, \frac{2\sin\frac{\nu\pi}{2}}{\nu\pi} \qquad \text{for} \quad \nu \quad \text{odd,}$$

$$n_{\nu M} = 0 \qquad \text{for} \quad \nu \quad \text{even,} \tag{III.3}$$

Thus the sequence of Fourier coefficients is

$$n_{\pm M} = \hat{n}\,\frac{2}{\pi},\ n_{\pm 3M} = -\,\hat{n}\,\frac{2}{3\pi},\ n_{\pm 5M} = \hat{n}\frac{2}{5\pi} \quad \text{etc..}$$

(b) basic structure with field free sections, according to fig.15(b) . If the length of the field free sections is Θ_s, we have to superpose on (III.3) a sequence of rectangular pulses of width Θ_s, and height $-\hat{n}$ starting at $\vartheta_0 = 0$, and sequence of height $\hat{n}$ starting at $\vartheta_0 = \frac{\Theta}{2}$, and find

$$n_{\nu M} = \hat{n}\frac{2}{\nu\pi}\left[\sin\frac{\nu\pi}{2} - \sin \nu\pi\,\frac{\Theta_s}{\Theta}\right] \quad \text{for} \quad \nu \quad \text{odd}$$

$$n_{\nu M} = 0 \qquad \text{for} \quad \nu \quad \text{even} \tag{III.4}$$

We note that $\frac{\Theta_s}{\Theta} = \frac{r_m - r_o}{2r_m} = \frac{1}{2}\,(1 - \frac{r_o}{r_m})$, because there are two straight sections per period Θ.

(ii) <u>The factor $\frac{r_o}{r_1} = \begin{matrix}1 \\ 0\end{matrix}$</u> $\begin{matrix}\text{in magnet sections} \\ \text{in straight sections}\end{matrix}$ has the Fourier components

$$\left(\frac{r_o}{r_1}\right)_0 = 1 - \frac{2\Theta_s}{\Theta} = \frac{r_o}{r_m}$$

$$\left(\frac{r_o}{r_1}\right)_{\nu M} = -\,\frac{\sin\, 2\pi\,\nu\frac{\Theta_s}{\Theta}}{\pi\nu} \qquad \text{for} \quad \nu \quad \text{even} \tag{III.5}$$

$$\left(\frac{r_o}{r_1}\right)_{\nu M} = 0 \qquad \text{for} \quad \nu \quad \text{odd}$$

(iii) <u>Field non-linearities</u> $\frac{dn}{dx}, \frac{d^2n}{dx^2}, \ldots$: If the alternation between $+|n|$ and $-|n|$ is produced by reversal of the same magnet pole profile, all even derivatives $\frac{d^2n}{dx^2}, \frac{d^4n}{dx^4}$, .. alternate with n whereas the odd derivatives $\frac{dn}{dx}, \frac{d^3n}{dx^3}, \ldots$ do not. This can be seen by making the transition $n(x) \to -\,n(-x)$ in the Taylor expansion of $n(x)$, and is illustrated by fig. 16 separately for a linear and a purely quadratic $n(x)$. Therefore $\frac{dn}{dx}$

varies with ϑ only because of field free sections (this can be taken account of by putting the factor $\frac{r_0}{r_1}$) whereas $\frac{d^2n}{dx^2}$ alternates like n, so that *(III.3) and (III.4) can be used to calculate the Fourier coefficients.*

(iv) Length corrections to magnet sectors : we assume that the magnet sectors are increased in length by $\Delta\Theta^F$ and $\Delta\Theta^D$ in the F-sectors and D-sectors respectively. The additional length shall be added to or subtracted from the end faces towards the straight sections according to the sign of $\Delta\Theta$. The increments $\Delta\Theta$ we have to envisage are so small that we can write(III.2):

$$g_{\nu P} = \frac{\hat{g}\Delta\Theta}{2\pi} \quad .$$

If g is the field index n, the incremental δn produced by the length corrections is represented by the Fourier coefficients

$$\delta n_{\nu M} = \frac{\hat{n}_F\ \Delta\vartheta^F}{2\pi}\, 2M \cos\frac{\nu M\vartheta_s}{2} + e^{i\nu M\frac{\vartheta}{2}}\ \frac{\hat{n}_D\Delta\vartheta^D}{2\pi}\, 2M \cos\frac{\nu M\vartheta_s}{2}$$

$$= \hat{n}(\Delta\Theta^F - \Delta\Theta^D\, e^{i\nu\pi})\, \frac{M}{\pi} \cos \nu\pi \frac{\Theta_s}{\Theta} \quad ,$$

because of $\hat{n}_D = -\hat{n}_F = -\hat{n}$, F-sectors starting at $\vartheta_0 = 0$. Thus

$$\left\{\begin{aligned} \delta n_{\nu M} &= \hat{n}(\Delta\Theta^F - \Delta\Theta^D)\, \frac{M}{\pi} \cos \nu\pi \frac{\Theta_s}{\Theta} \quad \text{for } \nu \text{ even} \\ \delta n_{\nu M} &= \hat{n}(\Delta\Theta^F + \Delta\Theta^D)\, \frac{M}{\pi} \cos \nu\pi \frac{\Theta_s}{\Theta} \quad \text{for } \nu \text{ odd.} \end{aligned}\right. \qquad \text{(III.6)}$$

If g stands for the field B, the incremental field δB produced has the Fourier coefficients

$$\left.\begin{aligned} \delta B_{\nu M} &= B(\Delta\Theta^F + \Delta\Theta^D)\, \frac{M}{\pi} \cos \nu\pi \frac{\Theta_s}{\Theta} \quad \text{for } \nu \text{ even} \\ \delta B_{\nu M} &= B(\Delta\Theta^F - \Delta\Theta^D)\, \frac{M}{\pi} \cos \nu\pi \frac{\Theta_s}{\Theta} \quad \text{for } \nu \text{ odd.} \end{aligned}\right\} \qquad \text{(III.7)}$$

as B does not change sign between F- and D-sectors.

For equal $\Delta\Theta^F$ and $\Delta\Theta^D$, only odd coefficients of δn and even coefficients of δB are non-zero.

(v) Lenses: For a set of S (= number of "superperiods") lenses producing a quantity $\hat{f}$ (standing for n, $\frac{dn}{dx}$, $\frac{d^2n}{dx^2}$, for quadrupole, sextupole, octupole lenses etc.) inside its longitudinal angular extension η, we obtain the Fourier coefficients

$$f_{\nu S} = \hat{f}\,\frac{\sin\frac{\nu S'\eta}{2}}{\nu\pi}\; e^{i\nu\frac{S}{N}m\pi}$$

$$f_0 = \hat{f}\,\frac{S\eta}{2\pi}$$

m is the number of the half-period at which the set starts. $2\frac{M}{S}$ different sets are possible, starting at $m = 0, 1, \ldots (2\frac{M}{S} - 1)$.

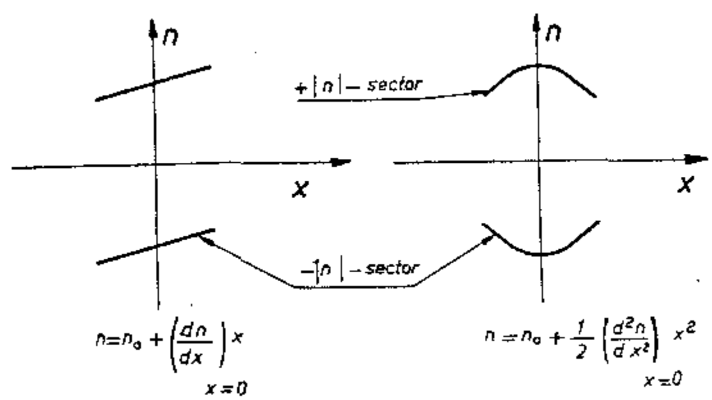

Fig. 16 Alternation and non-alternation of even and odd derivatives of n(x)

Appendix IV. Fourier analysis of Floquet factors

For the Floquet solution

$$x = w(\vartheta)\, e^{iQ\vartheta}$$

to solve the differential equation

$$\frac{d^2 x}{d\vartheta^2} - n(\vartheta)\, x = 0,$$

the differential equation

$$\frac{d^2 w}{d\vartheta^2} + 2iQ\,\frac{dw}{d\vartheta} - (Q^2 + n)w = 0 \qquad \text{(IV.1)}$$

must be satisfied by the Floquet factor $w(\vartheta)$. $w(\vartheta)$ is furthermore subject to the condition of periodicity with period $\Theta = \frac{2\pi}{M}$. Representing therefore $w(\vartheta)$ as well as $n(\vartheta)$ by Fourier series

$$w(\vartheta) = \sum_{\nu=-\infty}^{+\infty} w_{\nu M}\, e^{-i\nu M\vartheta}, \quad n(\vartheta) = \sum_{\nu} n_{\nu M}\, e^{-i\nu M\vartheta}, \qquad \text{(IV.2)}$$

and substituting these in (IV.1), an infinite set of linear equations is obtained for the Fourier coefficients $w_{\nu M}$:

$$w_{\nu M}(\nu M - Q)^2 + \sum_{\mu+\rho=\nu} n_{\mu M}\, w_{\rho M} = 0, \quad \nu = 0, \pm 1, \pm 2, \ldots \qquad \text{(IV.3)}$$

The sum is over all pairs μ, ρ with $\mu + \rho = \nu$. A non-zero solution $w(\vartheta)$ of this set of equations exists if Q is an appropriate eigenvalue, which must of course be the Q-value determined already by the methods of Appendix II. Q and the $n_{\mu M}$ being known, the equations (IV.3) lend themselves to quick solution by iterative approximation. Remembering from Appendix III, that only $n_{\mu M} = n_{-\mu M}$ with odd μ occur in the structures of interest here, we can write the equations for the first few Fourier coefficients

$$w_0 = -\frac{1}{Q^2}\left[n_M\,(w_M + w_{-M}) + n_{3M}(w_{3M} + w_{-3M}) + \ldots\right] \qquad \text{(IV.4)}$$

$$w_M = -\frac{1}{(M-Q)^2}\left[n_M(w_0 + w_{2M}) + n_{3M}\, w_{-2M} + \dots\right]$$

$$w_{-M} = -\frac{1}{(M+Q)^2}\left[n_M(w_0 + w_{-2M}) + n_{3M}\, w_{2M} + \dots\right]$$

$$w_{2M} = -\frac{1}{(2M-Q)^2}\left[n_M(w_M + w_{3M}) + n_{3M}\, w_{-M} + n_{5M}\, w_{-3M} + \dots\right]$$

$$w_{-2M} = -\frac{1}{(2M+Q)^2}\left[n_M(w_{-M} + w_{-3M}) + n_{3M}\, w_M + n_{5M}\, w_{3M} + \dots\right] \qquad \text{(VI.4)}$$

$$w_{3M} = \frac{1}{(2M+Q)^2}\left[n_M w_{2M} + n_{3M} w_0 + n_{5M}\, w_{-2M} + \dots\right]$$

$$w_{-3M} = -\frac{1}{(3M+Q)^2}\left[n_M w_{-2M} + n_{3M}\, w_0 + n_{5M}\, w_{2M} + \dots\right] .$$

As a common factor in all $w_{\nu M}$ is arbitrary we devide by w_0 and determine the ratios $\frac{w_M}{w_0}$, $\frac{w_{-M}}{w_0}$ etc. by iteration from the second and the following equations. If the period number M is large, convergence of the iteration is rapid.

The first equation

$$Q^2 = -\left[n_M \frac{w_M + w_{-M}}{w_0} + n_{3M} \frac{w_{3M} + w_{-3M}}{w_0} + \dots\right] \qquad \text{(IV.5)}$$

provides a check on the accuracy of the solution and on whether the right Q-value has been used.

The Floquet factor has to be normalized such as to make the determinant

$$i \begin{vmatrix} w_1(0) & \overset{*}{w}_1(0) \\ w_2(0) & \overset{*}{w}_2(0) \end{vmatrix} = i \begin{vmatrix} w(0) & \overset{*}{w}(0) \\ [iQw(0) + w'(0)] & [-iQ\overset{*}{w}(0) + \overset{*}{w}'(0)] \end{vmatrix} = 1$$

(see section 3). Introducing the Fourier series for $w(\vartheta)$ this condition becomes

$$2Qw_0^2\left(1 + \frac{w_M + w_{-M} + w_{2M} + w_{-2M} + w_{3M} + w_{-3M} + \dots}{w_0}\right) . \qquad \text{(IV.6)}$$

$$\left(1 + \frac{w_M + w_{-M} + w_{2M} + w_{-2M} + w_{3M} + w_{-3M} + \dots}{w_0} - \frac{M}{Q}\,\frac{w_M - w_{-M} + 2(w_{2M} - w_{-2M}) + 3(w_{3M} - w_{-3M}) + \dots}{w_0}\right) = 1$$

permitting the computation of the normalized w_0.

Numerical values of the Fourier coefficients of $n(\vartheta)$ and $w(\vartheta)$ obtained in this way for two structures close to the CERN P.S. are given in *the table below. Under "check", the r.h.s. of (IV.5), divided by* Q_1^2 is given, which should be equal to 1. Furthermore, under "structure (a), (IV.7), one finds the Fourier coefficients of $w(\vartheta)$ as computed from the following closed form expression which can be worked out from (II.12) in the case of structure (a) in a rather lengthy algebra :

$$w_{\nu M} = \frac{4n^{\frac{5}{4}}}{(\nu M-Q)^4 - n^2\sqrt{n}\Theta}\;\frac{1}{}\;\frac{(\nu M - Q)\,[1] - \sqrt{n}[2]}{[2\sin Q\Theta(\sinh\sqrt{n}\frac{\Theta}{2} + \sin\sqrt{n}\frac{\Theta}{2}\cosh\sqrt{n}\frac{\Theta}{2})]^{\frac{1}{2}}} \qquad \text{(IV.7)}$$

where [1], [2] are the expressions

$$[1] = \cos\sqrt{n}\frac{\Theta}{4}\,(\sinh\sqrt{n}\frac{\Theta}{2} + \sin\sqrt{n}\frac{\Theta}{2}\cosh\sqrt{n}\frac{\Theta}{2})(\sin\frac{\nu\pi}{2}\cos\frac{Q\Theta}{4} - \cos\frac{\nu\pi}{2}\sin\frac{Q\Theta}{4})$$

$$+ \sin\sqrt{n}\frac{\Theta}{4}\,\sin Q\Theta\,(\cos\frac{\nu\pi}{2}\cos\frac{Q\Theta}{4} + \sin\frac{\nu\pi}{2}\sin\frac{Q\Theta}{4})$$

$$[2] = \cos\sqrt{n}\frac{\Theta}{4}\,\sin Q\Theta\,(\sin\frac{\nu\pi}{2}\cos\frac{Q\Theta}{4} - \cos\frac{\nu\pi}{2}\sin\frac{Q\Theta}{4})$$

$$+ \sin\sqrt{n}\frac{\Theta}{4}\,(\sinh\sqrt{n}\frac{\Theta}{2} + \sin\sqrt{n}\frac{\Theta}{2}\cosh\sqrt{n}\frac{\Theta}{2})(\cos\frac{\nu\pi}{2}\cos\frac{Q\Theta}{4} + \sin\frac{\nu\pi}{2}\sin\frac{Q\Theta}{4}).$$

These $w_{\nu M}$ are already properly normalized. The comparison shows that the iteration method is perfectly satisfactory.

It is noteworthy that

$$w_0 \approx \left(\frac{1}{2Q}\right)^{\frac{1}{2}} = 0{,}283 \qquad \text{(IV.8)}$$

is a useful approximation (see (IV.6) and also section 3).

Table of Fourier coefficients of $n(\vartheta)$ and $w(\vartheta)$

$n(\vartheta)$	structure (a) (fig.15a)	structure (a) (IV.7)	structure (b) (fig.15 b)
M	50	50	50
number of straight sections(all equal)	0	0	100
$\frac{r_m}{r_o}$	1	1	$\frac{10\,000}{7\,008}$
Q	6¹/₄	6¹/₄	6¹/₄
$\hat{n}$	- 336.5	- 336.5	$-\ 282.4(\frac{r_m}{r_o})^2$
$n_{\pm M}$	- 214.2		$-\ 98,4(\frac{r_m}{r_o})^2$
$n_{\pm 3M}$	71.41		$119.0(\frac{r_m}{r_o})^2$
$n_{\pm 5M}$	- 42.84		$-\ 10.36(\frac{r_m}{r_o})^2$
w_o	0.287	0.289	0.287
$\frac{w_M}{w_o}$	0.1121	0.1122	0.1048
$\frac{w_{-M}}{w_o}$	0.0677	0.0678	0.0632
$\frac{w_{2M}}{w_o}$	0.00208	0.00207	0.00035
$\frac{w_{-2M}}{w_o}$	0.00051	0.00050	- 0.00133
$\frac{w_{3M}}{w_o}$	- 0.00343	- 0.00344	- 0.01173
$\frac{w_{-3M}}{w_o}$	- 0.00292	- 0.00292	- 0.00994
check	0.9975	1.0016	0.9959

The Floquet factor for z is according to (II.16) :

$$u(\vartheta) \equiv u_1(\vartheta) = w(\vartheta + \frac{\Theta}{2}) = \sum_\nu w_{\nu M} \, e^{-i(\nu M\vartheta + \frac{\nu M\Theta}{2})}$$

$$= \sum_\nu w_{\nu M} \, e^{-i\nu\pi} \, e^{-i\nu M\vartheta}$$

$$= \sum_\nu u_{\nu M} \, e^{-i\nu M\vartheta}$$

Thus its Fourier components are simply

$$u_{\nu M} = (-1)^\nu \, w_{\nu M} \qquad \text{(IV.9)}$$

Appendix V. Perturbations of the closed orbit

In the transformation (8.8) to "smooth motion" coordinates we substitute the Fourier series for the Floquet factors, and obtain for the transformation coefficients :

$$\sqrt{\frac{Q}{2}}\,(w_1 + w_1^*) = w_0\sqrt{2Q}\;[1 + \rho_1 \cos M\Theta + \rho_2 \cos 2M\vartheta + ..]$$

$$\sqrt{\frac{Q}{2}}\,\frac{w_1 - w_1^*}{iQ} = \frac{w_0\sqrt{2Q}}{Q}\;[-\sigma_1 \sin M\vartheta - \sigma_2 \sin 2M\vartheta - ...]$$

$$\sqrt{\frac{Q}{2}}\,(w_2 + w_2^*) = w_0\sqrt{2Q}\;[(Q\sigma_1 - M\rho_1)\sin M\vartheta + (Q\sigma_2 - M\rho_2)\sin 2M\vartheta + ..]$$

$$\sqrt{\frac{Q}{2}}\,\frac{w_2 - w_2^*}{iQ} = \frac{w_0\sqrt{2Q}}{Q}\;[Q + (Q\rho_1 - M\sigma_1)\cos M\vartheta + (Q\rho_2 - M\sigma_2)\cos 2M\vartheta + ..]$$

where

$$\rho_1 = \frac{w_M + w_{-M}}{w_0}\;,\quad \rho_2 = \frac{w_{2M} + w_{-2M}}{w_0}\;,\;.. \qquad \text{(V.1)}$$
$$\sigma_1 = \frac{w_M - w_{-M}}{w_0}\;,\quad \sigma_2 = \frac{w_{2M} - w_{-2M}}{w_0}\;,\;..$$

Using the approximation (IV.8)

$$w_0 \approx \frac{1}{\sqrt{2Q}}\;,$$

which is very good in the cases of interest here, we can write the transformation (8.8)

$$x = (1 + \rho_1 \cos M\vartheta + ..)\,\bar{x} - \frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{x}' \qquad \text{(V.2)}$$
$$x' = Q\left[(\sigma_1 - \frac{M}{Q}\rho_1)\sin M\vartheta + ..\right]\bar{x} + \left[1 + (\rho_1 - \frac{M}{Q}\sigma_1)\cos M\vartheta + ..\right]\bar{x}'$$

The corresponding transformation for the z-motion to smooth motion variables $\bar{z}$, $\bar{z}'$ is obtained from (V.2) by changing ρ_ν, σ_ν into $(-1)^\nu\rho_\nu$, $(-1)^\nu\sigma_\nu$ as follows from the property (IV.9) of the Floquet factors.

When introducing (V.2) into the perturbation Hamiltonian (I.24)

$$H^{(1)} = V_{10}x + V_{01}z + V_{20}x^2 + V_{02}z^2 + V_{11}xz + V_3(x^3 - 3xz^2) + V_4(x^4 - 6x^2z^2 + z^4) + ..$$
$$+ \frac{1}{2r_1}\,x[(x')^2 + (z')^2] \qquad \text{(V.3)}$$

we neglect squares and higher powers of small quantities on account of the fact that $\rho_\nu \ll 1$, $\sigma_\nu \ll 1$. e.g. in the case of the CERN PS the numerical values (following from the table of Appendix IV) are

$$\left[\begin{array}{l} \rho_1 = 0.168 \qquad \rho_2 = -0.0010 \qquad \rho_3 = -0.0217 \\ \sigma_1 = 0.0416 \qquad \sigma_2 = 0.0017 \qquad \sigma_3 = -0.0018 \\ \\ \frac{\sigma_1}{Q} = 0.00667 \\ (\sigma_1 - \frac{M}{Q}\rho_1) = -1.304; \; (\sigma_2 - \frac{M}{Q}\rho_2) = 0.0097; \; (\sigma_3 - \frac{M}{Q}\rho_3) = 0.175 \\ \rho_1 - \frac{M}{Q}\sigma_1 = -0.165; \; \rho_2 - \frac{M}{Q}\sigma_2 = -0.0146; \; \rho_3 - \frac{M}{Q}\sigma_3 = -0.0073. \end{array}\right. \qquad (V.4)$$

The perturbation Hamiltonian then becomes

$$\begin{aligned} H^{(1)} = & V_{10}\left([1 + \rho_1 \cos M\vartheta + ..]\,\bar{x} - \frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{x}'\right) \\ & + V_{01}\left([1 - \rho_1 \cos M\vartheta + ..]\,\bar{z} + \frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{z}'\right) \\ & + V_{20}\left([1 + 2\rho_1 \cos M\vartheta + ..]\,\bar{x}^2 - 2\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{x}\,\bar{x}'\right) \\ & + V_{02}\left([1 - 2\rho_1 \cos M\vartheta + ..]\,\bar{z}^2 + 2\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{z}\,\bar{z}'\right) \\ & + V_{11}\left(\bar{x}\,\bar{z} + \frac{\sigma_1 \sin M\vartheta + ..}{Q}\,(\bar{x}\,\bar{z}' - \bar{x}'\bar{z})\right) \\ & + V_3\left([1 + 3\rho_1 \cos M\vartheta + ..]\,\bar{x}^3 - 3\,\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{x}^2\,\bar{x}'\right. \\ & \left. - 3[1 - \rho_1 \cos M\vartheta + ..]\,\bar{x}\,\bar{z}^2 - 3\,\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,(2\,\bar{x}\,\bar{z}\,\bar{z}' - \bar{x}'\bar{z}^2)\right) \\ & + V_4\left([1 + 4\rho_1 \cos M\vartheta + ..]\,\bar{x}^4 - 4\,\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{x}^3\bar{x}'\right. \\ & - 6\,\bar{x}^2\bar{z}^2 + 12\,\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,(\bar{x}\,\bar{x}'\bar{z}^2 - \bar{x}^2\bar{z}\,\bar{z}') \\ & \left. + [1 - 4\rho_1 \cos M\vartheta + ..]\,\bar{z}^4 + 4\,\frac{\sigma_1 \sin M\vartheta + ..}{Q}\,\bar{z}^3\bar{z}'\right) \\ & + \ldots \end{aligned}$$

$$+ \frac{1}{2r_1}\left([1+\rho_1\cos M\vartheta+..]\bar{x} - \frac{\sigma_1\sin M\vartheta+..}{Q}\bar{x}'\right)\Big\{Q^2[(\sigma_1-\frac{M}{Q}\rho_1)\sin M\vartheta+ ..]^2(\bar{x}^2+\bar{z}^2)$$

$$+ (\bar{x}')^2+ (\bar{z}')^2+ 2[(\rho_1-\frac{M}{Q}\sigma_1)\cos M\vartheta+..]\left((\bar{x}')^2- (\bar{z}')^2\right)$$

$$+ 2Q\ [(\sigma_1-\frac{M}{Q}\rho_1)\sin M\vartheta+..]\ (\bar{x}\ \bar{x}' - \bar{z}\ \bar{z}')$$

$$+ 2Q[(\sigma_1-\frac{M}{Q}\rho_1)\sin M\vartheta+..]\ [(\rho_1- \frac{M}{Q}\sigma_1)\ \cos M\vartheta\ +..](\bar{x}\ \bar{x}'+ \bar{z}\ \bar{z}')\Big\}. \qquad (V.5)$$

We now make a further approximation by neglecting those terms of $H^{(1)}$ the coefficients of which have frequencies much higher than Q. This is justified by the form of equation (8.9), respectively(V.9,10) below : right hand side terms oscillating with a frequency much higher than Q contribute little to the closed orbit solution. In the CERN PS $M = 50$ is large enough in comparison with $Q = 6^{1/4}$ to permit omission of all terms showing the periodicity of the A.G. structure. The only terms to be kept are therefore those with constant or low order harmonic coefficients. The latter can arise either from a superperiodicity introduced by lenses, or from accidental structure deficiencies.

First we consider the case of systematic perturbations only. Defining Fourier coefficients of V_{10}, V_{01}, ... by

$$V_{k_1k_2}(\vartheta) = \sum_{q=-\infty}^{+\infty} V_{k_1k_2q}\ e^{-iq\vartheta} \qquad (V.6)$$

only the harmonics $q = 0, \pm M, \pm 2M, ..$ appear in systematic perturbations.

In this case furthermore

$$V_{k_1k_2(-\nu M)} = V_{k_1k_2(\nu M)}$$

(See Appendix III and VI), therefore only $\cos \nu M\vartheta$ - terms occur in the Fourier series (V.6). The "low frequency" part of $H^{(1)}$ then is found to be

$$\overline{H^{(1)}} = [V_{10,0} + \rho_1\ V_{10,M} + \rho_2\ V_{10,2M} + \rho_3\ V_{10,3M} + ..]\ \bar{x}$$

$$+ [V_{20,0} + 2\rho_1\ V_{20,M} + 2\rho_2\ V_{20,2M} + 2\rho_3\ V_{20,3M} + ..]\ \bar{x}^2$$

$$+ [V_{02,0} - 2\rho_1\ V_{02,M} + 2\rho_2\ V_{02,2M} - 2\rho_3\ V_{02,3M} + ..]\ \bar{z}^2$$

$$+ V_{30,0}\,(\bar{x}^3 - 3\,\bar{x}\,\bar{z}^2) + 3\rho_1\, V_{30,M}\,(\bar{x}^3 + 3\,\bar{x}\,\bar{z}^2)$$

$$+ V_{40,0}\,(\bar{x}^4 - 6\,\bar{x}^2\bar{z}^2 + \bar{z}^4) + 4\rho_1\, V_{40,M}\,(\bar{x}^4 - \bar{z}^4)$$

$$+ \ldots$$

$$+ \frac{1}{2r_m}\,\bar{x}\left((\bar{x}')^2 + (\bar{z}')^2 + \frac{Q^2}{2}\,(\sigma_1 - \frac{M}{Q}\,\rho_1)^2\,(\bar{x}^2 + \bar{z}^2)\right) \qquad (V.7)$$

The low frequency part of the momentum dependent term (last line of (V.7)) would in general be somewhat more complicated. The neglections in this part have been made on the basis of the numerical values of the coefficients holding for the CERN PS. In this particular case the neglected terms do not exceed a few percent of those written down in the last line of (V.7).

Specializing the Hamiltonian to systematic perturbations due to momentum deviations, fringing fields and n-deviations by taking the pertaining coefficients $V_{k_1k_2(\nu M)}$ from Appendix (I.24) it reads

$$H^{(1)} \doteq \left(\frac{r_m}{r_o}\right)^2 \frac{p_o}{p} \Bigg\{ \left[-\frac{r_o^2}{r_m}\frac{\delta p}{p_o} + r_o \left(\frac{\delta B_z}{B_o}\right)_o + \rho_1\, r_o \left(\frac{\delta B_z}{B_o}\right)_M + \ldots \right] \bar{x}$$

$$+ \frac{1}{2}\left[\frac{r_o}{r_m} + 2(\rho_1 n_M + \rho_3 n_{3M} + \ldots)\,\frac{\delta p}{p_o} - (\delta n_o + 2\rho_1\,\delta n_M + 2\rho_2\,\delta n_{2M} + \ldots)\right] \bar{x}^2$$

$$- \frac{1}{2}\left[\; - 2(\rho_1 n_M + \rho_3 n_{3M} + \ldots)\,\frac{\delta p}{p_o} - (\delta n_o - 2\rho_1\,\delta n_M + 2\rho_2\,\delta n_{2M} - \ldots)\right] \bar{z}^2$$

$$- \frac{1}{3!}\left[\frac{dn}{dx}\right]_o (\bar{x}^3 - 3\bar{x}\,\bar{z}^2) \qquad - \frac{1}{4!}\left[\frac{d^2 n}{dx^2}\right]_M 4\rho_1\,(\bar{x}^4 - \bar{z}^4) - \ldots \Bigg\}$$

$$+ \frac{1}{2r_m}\,\bar{x}\left\{(\bar{x}')^2 + (\bar{z}')^2 + \frac{Q^2}{2}(\sigma_1 - \frac{M}{Q}\,\rho_1)^2\,(\bar{x}^2 + \bar{z}^2)\right\} \qquad (V.8)$$

Here δp, δB_z, δn are the deviations from the nominal "unperturbed" values, the suffixes indicating the harmonic numbers of their Fourier components. Expressed in terms of equivalent length corrections and magnet - n - values they will be found in Appendix VI. With (V.8) we obtain the equations of motion (8.9) (in two dimensions) for systematic perturbations :

$$\frac{d^2\bar{x}}{d\vartheta^2} + Q^2\bar{x} = -\frac{\partial H^{(1)}}{\partial\bar{x}} + \frac{d}{d\vartheta}\frac{\partial H^{(1)}}{\partial\bar{x}'}$$

$$= \frac{p_o}{p}\Bigg\{ r_m\frac{\delta p}{p_o} - \frac{r_m}{r_o} r_m \left[\left(\frac{\delta B_z}{B_o}\right)_o + \rho_1\left(\frac{\delta B_z}{B_o}\right)_M + ..\right]$$

$$-\left(\frac{r_m}{r_o}\right)^2\left[\frac{r_o}{r_m} + 2(\rho_1\ n_M + \rho_3\ n_{3M})\frac{\delta p}{p_o} - (\delta n_o + 2\rho_1\ \delta n_M + 2\rho_2\ \delta n_{2M}..)\right]\bar{x}$$

$$+\frac{1}{2!}\left(\frac{r_m}{r_o}\right)^2\left[\frac{dn}{dx}\right]_o(\bar{x}^2 - \bar{z}^2) + ...\Bigg\} \qquad (V.9)$$

$$\frac{d^2\bar{z}}{d\vartheta^2} + Q^2\bar{z} = -\frac{\partial H^{(1)}}{\partial\bar{z}} + \frac{d}{d\vartheta}\frac{\partial H^{(1)}}{\partial\bar{z}'}$$

$$= \left(\frac{r_m}{r_o}\right)^2\frac{p_o}{p}\Bigg\{\left[-(\rho_1\ n_M + \rho_3\ n_M)\frac{\delta p}{p_o} - (\delta n_o - \rho_1\ \delta n_M + \rho_2\ \delta n_{2M} - ..)\right]\bar{z}$$

$$-\frac{1}{2!}\left[\frac{dn}{dx}\right]_o 2\ \bar{x}\ \bar{z} - ...\Bigg\} \qquad (V.10)$$

The first line on the r.h.s. of (V.9) causes the radial shift of the closed orbit by momentum deviation and guide field deviations. The second line in (V.9), respectively first line in (V.10) produce shifts of the frequencies by

$$\delta Q_1 = \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2\frac{p_o}{p}\left[\frac{r_o}{r_m} + 2(\rho_1\ n_M + \rho_3\ n_{3M} + ..)\frac{\delta p}{p_o} - (\delta n_o + 2\rho_1\ \delta n_M + 2\rho_2\ \delta n_{2M} + ..)\right]$$

$$= \frac{p_o}{p}\left[\frac{1}{2Q}\frac{r_m}{r_o} - Q\frac{\delta p}{p_o} - \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2(\delta n_o + 2\rho_1\ \delta n_M + ..)\right]$$

$$\delta Q_2 = \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2\frac{p_o}{p}\left[2(\rho_1 n_M + \rho_3\ n_{3M} + ..)\frac{\delta p}{p_o} + (\delta n_o - 2\rho_1\ \delta n_M + 2\rho_2\ \delta n_{2M} - ..)\right]$$

$$= \frac{p_o}{p}\left[-Q\frac{\delta p}{p_o} + \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2(\delta n_o - 2\rho_1\ \delta n_M + ...)\right] \qquad (V.11)$$

where use has been made of (IV.5)

$$\left(\frac{r_m}{r_o}\right)^2(\rho_1\ n_M + \rho_3\ n_{3M} + ..) = -Q^2 \qquad (V.12)$$

If δn represents only a correction of the magnet-n-value, the last terms can correspondingly be replaced by

$$\frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2(2\rho_1\ \delta n_M + 2\rho_3\ \delta n_{3M} + ..) = -\frac{\delta\hat{n}}{\hat{n}}Q.$$

Similarly

$$\left(\frac{r_m}{r_o}\right)^2 \left[\rho_1 \left(\frac{d^2 n}{dx^2}\right)_M + \,..\right] = -\frac{Q^2}{\hat{n}} \frac{d^2 \hat{n}}{dx^2}$$

The expressions (V.11) agree with those obtained by the method of variation of phase and amplitude to the degree of approximation used. (See Appendix VI for comparison).

Equations (V.9)...(V.11) may serve for calculating the modifications of sector lengths and n-values necessary to bring back the closed orbit, Q_1, and Q_2 *to the nominal values. Introducing length adjustments* $\Delta\ell^F$, $\Delta\ell^D$ of F- and D-sectors, the equivalent length $\Delta\ell$ of the fringing field per magnet edge (facing a field free section; see Appendix I), and an adjustment $\delta\hat{n}$ of the magnet - n - value from Appendix VI, we have to require

$$r_o \frac{\delta p}{p_o} - \frac{M}{\pi} (\Delta\ell^F + \Delta\ell^D + 2\Delta\ell) \left[1 + \rho_2 \cos 2\frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right) + \,..\right]$$
$$- \frac{M}{\pi} (\Delta\ell^F - \Delta\ell^D) \left[\rho_1 \cos \frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right) + \rho_3 \cos 3\frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right) + \,..\right] = 0 \qquad \text{(V.13)}$$

$$\frac{p}{p_o}\delta Q_1 = \frac{r_m}{r_o}\frac{1}{2Q} - \left(\frac{\delta p}{p_o} - \frac{\delta\hat{n}}{\hat{n}}\right) Q - \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2 \left\{ \hat{n}\frac{M}{\pi} \frac{\Delta\ell^F - \Delta\ell^D}{r_m}\left[1 + 2\rho_2 \cos 2\frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right)\right]\right.$$
$$\left. + \hat{n}\frac{M}{\pi} \frac{\Delta\ell^F + \Delta\ell^D + 2\Delta\ell - 2r_o \frac{d\Delta\ell}{dx}}{r_m} 2\left[\rho_1 \cos\frac{\pi}{2}\left(1-\frac{r_o}{r_m}\right) + \rho_3 \cos 3\frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right)\right]\right\} = 0 \qquad \text{(V.14)}$$

$$\frac{p}{p_o}\delta Q_2 = -\left(\frac{\delta p}{p_o} - \frac{\delta\hat{n}}{\hat{n}}\right) Q + \frac{1}{2Q}\left(\frac{r_m}{r_o}\right)^2 \left\{ \hat{n}\frac{M}{\pi} \frac{\Delta\ell^F - \Delta\ell^D}{r_m}\left[1 + 2\rho_2 \cos 2\frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right)\right]\right.$$
$$\left. - \hat{n}\frac{M}{\pi} \frac{\Delta\ell^F + \Delta\ell^D + 2\Delta\ell - 2r_o \frac{d\Delta\ell}{dx}}{r_m} 2\left[\rho_1 \cos\frac{\pi}{2}\left(1-\frac{r_o}{r_m}\right) + \rho_3 \cos 3\frac{\pi}{2}\left(1 - \frac{r_o}{r_m}\right)\right]\right\} = 0$$

Imposing the supplementary condition $\frac{\delta p}{p_o} = 0$, the corrections $\Delta\ell^F$, $\Delta\ell^D$, and and $\delta\hat{n}$ are determined by these equations. Using CERN PS data (see section 15 and Appendix I) one finds $\Delta\ell^F = -\Delta\ell - 0{,}33$ cm, $\Delta\ell^D = -\Delta\ell + 0.45$ cm, $\frac{\delta\hat{n}}{\hat{n}} = 0.016$.

Supposing that adjustments according to (V.13) (V.14), and (V.15) have been made to bring the equilibrium orbit and the Q-values back to their nominal positions, the equations (V.9) (V.10) can be simplified :

$$\frac{d^2\bar{x}}{d\vartheta^2} + Q^2\bar{x} = \frac{p_0}{p}\left\{r_m \frac{\delta p}{p_0} + 2Q^2 \frac{\delta p}{p_0}\bar{x} + \frac{1}{2!}\left(\frac{r_m}{r_0}\right)^2\left[\frac{dn}{dx}\right]_0 (\bar{x}^2 - \bar{z}^2) + \ldots\right\}$$

$$\frac{1}{2r_m}\left[(\bar{x}')^2 - (\bar{z}')^2 - \frac{Q^2}{2}(\sigma_1 - \frac{M}{Q}\rho_1)^2(3\bar{x}^2 + \bar{z}^2)\right] + \frac{1}{r_m}\bar{x}\frac{d^2\bar{x}}{d\vartheta^2} \qquad (V.16)$$

$$\frac{d^2\bar{z}}{d\vartheta^2} + Q^2\bar{z} = + \frac{p_0}{p}\left\{2\,Q^2 \frac{\delta p}{p_0}\bar{z} - \frac{1}{2!}\left(\frac{r_m}{r_0}\right)^2 \left[\frac{dn}{dx}\right]_0 2\,\bar{x}\,\bar{z} - \ldots\right\}$$

$$- \frac{Q^2}{2r_m}(\sigma_1 - \frac{M}{Q}\rho_1)^2\,\bar{x}\,\bar{z} + \frac{1}{r_m}\left(\bar{x}'\,\bar{z}' + \bar{x}\frac{d^2\bar{z}}{d\vartheta^2}\right) \qquad (V.17)$$

where δp is now the deviation from the adjusted equilibrium momentum. *From (V.16), the* *radial displacement* $\overline{\delta r} = \bar{c}$ *of the closed orbit provoked by* δp follows. In the absence of non-linearities one obtains to first approximation in $\frac{\delta p}{p_0}$

$$\frac{\overline{\delta r}}{r_m} = \frac{\bar{c}}{r_m} = \frac{1}{Q^2}\frac{\delta p}{p_0}\,. \qquad (V.18)$$

This relation shows that the "momentum compaction"factor $\frac{\delta p/p_0}{\delta r/r_0}$ is equal to Q^2. *$\overline{\delta r} = \bar{c}$ is independent of ϑ and represents only the "smooth"* part of the displaced closed orbit. Its real shape is obtained by multiplying by the "wriggling" factor from (V.2)

$$c(\vartheta) = \bar{c}(1 + \rho_1 \cos M\vartheta + \ldots). \qquad (V.19)$$

If the non-linear terms in (V.16) and the difference between $\frac{\delta p}{p_0}$ and $\frac{\delta p}{p}$ are taken into account, the following more correct relation then emerges for the displacement $\overline{\delta r} = \bar{c}$ of the closed orbit

$$Q^2\bar{c} - \frac{1}{2}\left[\left(\frac{r_m}{r_0}\right)^2 \left(\frac{dn}{dx}\right)_0 - \frac{3}{2}\frac{Q^2}{r_m}(\sigma_1 - \frac{M}{Q}\rho_1)^2\right]\bar{c}^2 + \ldots$$

$$= \left[r_m + Q^2\bar{c} - \frac{3}{4}\frac{Q^2}{r_m}(\sigma_1 - \frac{M}{Q}\rho_1)^2\,\bar{c}^2\right]\frac{\delta p}{p_0}$$

When looking at numerical values of the coefficients involved the third term on the right hand side is found to be negligible. Solving then for $\frac{\delta p}{p_o}$, *the relation can be written*

$$\frac{\delta p}{p_o} = Q^2 \frac{\bar{c}}{r_m} \left\{ 1 - \left[Q^2 + \frac{1}{2} \left(\frac{r_m}{r_o}\right)^2 \left(\frac{dn}{dx}\right)_o \frac{r_m}{Q^2} - \frac{3}{2} \left(\sigma_1 - \frac{M}{Q} \rho_1\right)^2 \right] \frac{\bar{c}}{r_m} + \ldots \right. \qquad (V.20)$$

Putting in the numbers for the CERN PS we obtain

$$\frac{\delta p}{p_o} = 39 \frac{\overline{dr}}{r_m} \left\{ 1 - \left[39 + 250 \left(\frac{dn}{dx}\right)_o - 2.5\right] \frac{\overline{\delta r}}{r_m} + \ldots \right\}$$

$\frac{\overline{\delta r}}{r_m} < 10^{-3}$ within the limits of the vacuum chamber. For $\left(\frac{dn}{dx}\right)_o$ values up to the order of 1 cm^{-1} may be considered. For a number of particular cases the relation between $\overline{dr}$ and $\frac{\delta p}{p_o}$ is shown graphically in figs. 17-22 (p.126-128)

In treating now the <u>*effects of accidental perturbations on the*</u> <u>closed orbit</u>, we disregard -- for the sake of simplicity -- irrelevant systematic perturbations like fringing fields, as well as perturbations due to unsystematic irregularities of n, $\frac{dn}{dx}$, $\frac{d^2n}{dx^2}$ etc. The perturbation Hamiltonian (V.3) is then, written explicitely with the help of (I.24)

$$\begin{aligned} H^{(1)} = \left(\frac{r_m}{r_o}\right)^2 \frac{p_o}{p} \Bigg\{ & \left[-\frac{r_o^2}{r_1} \frac{\delta p}{p_o} + r_o \frac{\delta B_z(\vartheta)}{B_o} + n(\vartheta)\xi(\vartheta) \right] x + \left[\varepsilon(\vartheta) r_o - n(\vartheta)\zeta(\vartheta)\right] z \\ & + \frac{1}{2} n(\vartheta) \frac{\delta p}{p_o} (x^2 - z^2) - 2\varepsilon(\vartheta) n(\vartheta) xz \\ & - \frac{1}{3!} \frac{dn}{dx} (x^3 - 3xz^2) - \frac{1}{4!} \frac{d^2n}{dx^2} (x^4 - 6x^2z^2 + z^4) - \ldots \Bigg\} \\ & + \frac{1}{2r_1} x \left[(x')^2 + (z')^2\right] \end{aligned}$$

$\delta B_z(\vartheta)$ represents the field fluctuation along the magnet axis; ξ, ζ, ϵ the radial and vertical displacements, and the angle by which the plane of symmetry of the magnets is twisted. Working out the "low frequency" part of this Hamiltonian, after transformation to the variables $\bar{x}$, $\bar{z}$, $\bar{x}'$, $\bar{z}'$ as given by (V.5), results in

$$H^{(1)} = - r_m \frac{\delta p}{p} \bar{x} - Q^2 \frac{\delta p}{p} (\bar{x}^2 + \bar{z}^2) +$$

$$+ (\frac{r_m}{r_o})^2 \frac{p_o}{p} \left\{ \left[r_o \frac{\delta B_z(\vartheta)}{B_o} + n(\vartheta)\xi(\vartheta) + (\rho_1 n_M + \rho_3 n_{3M} + ..)\xi(\vartheta) \right] \bar{x} \right.$$

$$+ \left[r_o\epsilon(\vartheta) - n(\vartheta)\zeta(\vartheta) + (\rho_1 n_M + \rho_3 n_{3M} + ..)\zeta(\vartheta)\right] \bar{z} - 2\epsilon(\vartheta)n(\vartheta)\bar{x}\bar{z}$$

$$\left. - \frac{1}{3!}(\frac{dn}{dx})_o(\bar{x}^3 - 3\bar{x}\bar{z}^2) + \frac{1}{4}(\frac{d^2n}{dx^2})_o(\bar{x}^4 - 6\bar{x}^2\bar{z}^2 + \bar{z}^4) + \frac{4}{4!}\left[\rho_1(\frac{d^2n}{dx^2})_M + ..\right](\bar{x}^4 - \bar{z}^4)..\right\}$$

$$+ \frac{1}{2r_m} \bar{x} \left[(\bar{x}')^2 + (\bar{z}')^2 + \frac{Q^2}{2}(\sigma_1 - \frac{M}{Q}\rho_1)^2(\bar{x}^2 + \bar{z}^2) \right] + .. \qquad (V.21)$$

From (V.21) we obtain the equations of motion

$$\frac{d^2\bar{x}}{d\vartheta^2} + Q^2\bar{x} = - \frac{\partial H^{(1)}}{\partial \bar{x}} =$$

$$= \frac{p_o}{p} \left\{ r_m \frac{\delta p}{p_o} - (\frac{r_m}{r_o})^2 \left[r_o \frac{\delta B_z(\vartheta)}{B_o} + n(\vartheta)\xi(\vartheta) \right] + Q^2\xi(\vartheta) + 2Q^2\frac{\delta p}{p_o}\bar{x} + 2\epsilon(\vartheta)n(\vartheta)\bar{z} \right.$$

$$\left. + (\frac{r_m}{r_o})^2 \left[\frac{1}{2}(\frac{dn}{dn})_o(\bar{x}^2 - \bar{z}^2) + \frac{1}{3!}\left((\frac{d^2n}{dx^2})_o + 4\rho_1(\frac{d^2n}{dx^2})_M + .. \right)\bar{x}^3 - \frac{3}{3!}(\frac{d^2n}{dx^2})_o\bar{x}\bar{z}^2 + .. \right] \right\}$$

(V.22)

$$\frac{d^2\bar{z}}{d\vartheta^2} + Q^2\bar{z} = - \frac{\partial H^{(1)}}{\partial \bar{z}} =$$

$$= \frac{p_o}{p} \left\{ -(\frac{r_m}{r_o})^2[r_o\epsilon(\vartheta) - n(\vartheta)\zeta(\vartheta)] + Q^2\zeta(\vartheta) + 2Q^2 \frac{\delta p}{p_o} \bar{z} + 2\epsilon(\vartheta)n(\vartheta) \bar{x} \right.$$

$$\left. + (\frac{r_m}{r_o})^2 \left[- \frac{1}{2}(\frac{dn}{dx})_o 2\bar{x}\bar{z} + \frac{1}{3!}\left((\frac{d^2n}{dx^2})_o - 4\rho_1(\frac{d^2n}{dx^2})_M - .. \right)\bar{z}^3 - \frac{3}{3!}(\frac{d^2n}{dx^2})_o\bar{x}^2\bar{z} + .. \right] \right\}$$

(V.23)

The terms depending on $\bar{x}'$ and $\bar{z}'$ have been omitted here; unless the nonlinearities of the field are very small, their relative influence is negligible in the case of the CERN PS as shown by the figures introduced into (V.20) above.

The forcing terms $Q^2\xi$ and $Q^2\zeta$ appearing in the equations (V.22), (V.23) correspond to what one would expect for an oscillator with displaced equilibrium position. There are, however, additional forcing terms $n\xi$ and $n\zeta$ produced by magnet displacements. If ξ and ζ would only smoothly vary with ϑ, these terms would alternate as $n(\vartheta)$ and therefore be "high frequency", and have little effect on $\bar{x}$. For random displacements, however, $n(\vartheta)\xi(\vartheta)$ may give rise to appreciable low frequency contributions.

In the CERN PS one half F-sector and one-half D-sector are always combined to form a rigid magnet unit. In this way the force $n(\vartheta)\xi(\vartheta)$ almost cancels if a unit is displaced transversely parallel to itself. The effect of a tilt of the unit is of course not eliminated by this. A similar statement holds for the linear coupling terms $2\,\epsilon(\vartheta)\,n\,(\vartheta)\,\bar{x}$ and $2\,\epsilon(\vartheta)\,n\,(\vartheta)\,\bar{z}$: The effect of ϵn due to a rigid rotation ϵ of a whole $\frac{1}{2}$ F $\frac{1}{2}$ D unit about the orbit axis almost cancels, not, however, the effect due to a twisted unit.

In calculating the Fourier components of the forcing terms $n(\vartheta)\,\xi(\vartheta)$ and $n(\vartheta)\zeta(\vartheta)$, for $\frac{1}{2}$ F $\frac{1}{2}$ D sectors combined to form rigid magnet units, we assume that the unit centers are positioned at (see fig. 15b)

$$\vartheta_k = k\,\frac{\Theta}{2} - \frac{\Theta}{4} = (k - \tfrac{1}{2})\,\frac{\pi}{M}\,,$$

and extend from

$$\vartheta_k - \frac{\eta}{2} \quad \text{to} \quad \vartheta_k + \frac{\eta}{2}\,,$$

where

$$\eta = \frac{\Theta}{2}\,\frac{r_o}{r_m} = \frac{\pi}{M}\,\frac{r_o}{r_m}$$

the angular length of a unit. The Fourier coefficients are now

$$(n\xi)_q = \frac{1}{2\pi}\int_0^{2\pi} n\xi e^{iq\Theta}\,d\vartheta = \frac{1}{2\pi}\sum_{k=1}^{2M}\int_{\vartheta_k-\frac{\eta}{2}}^{\vartheta_k+\frac{\eta}{2}} n\left[\xi(\vartheta_k)+\frac{(\vartheta-\vartheta_k)}{\eta}\,\Delta\xi_k\right]e^{iq\vartheta}d\vartheta$$

$\Delta\xi_k$ is the difference in radial displacement of the ends of the k-th magnet unit due to tilt in the horizontal plane. Denoting by $\hat{n}$ the value of

$n(\vartheta)$ in the sectors adjacent to $\vartheta = 0$ (as in Appendix III) this becomes

$$(n\xi)_q = \sum_{k=1}^{2M} \frac{\xi(\vartheta_k)\ \hat{n}\ (-1)^k}{2\pi} \left[- \int_{\vartheta_k - \frac{\eta}{2}}^{\vartheta_k} e^{iq\vartheta}\, d\vartheta + \int_{\vartheta_k}^{\vartheta_k + \frac{\eta}{2}} e^{iq\vartheta}\, d\vartheta \right]$$

$$+ \sum_{k=1}^{2M} \frac{\Delta\xi_k\ \hat{n}\ (-1)^k}{2\pi\eta} \left[- \int_{\vartheta_k - \frac{\eta}{2}}^{\vartheta_k} (\vartheta - \vartheta_k)\, e^{iq\vartheta}\, d\vartheta + \int_{\vartheta_k}^{\vartheta_k + \frac{\eta}{2}} (\vartheta - \vartheta_k) e^{iq\vartheta}\, d\vartheta \right]$$

$$= \frac{\hat{n}}{\pi q}\ e^{-i\frac{q\pi}{2M}} \left\{ i\left[1 - \cos\frac{q\eta}{2}\right] \sum_{k=1}^{2M} (-1)^k\, \xi(\vartheta_k)\ e^{ik\frac{q2\pi}{2M}} \right.$$

$$\left. + \frac{1}{2}\left[\sin\frac{q\eta}{2} - \frac{1 - \cos\frac{q\eta}{2}}{\frac{q\eta}{2}} \right] \sum_{k=1}^{2M} (-1)^k\ \Delta\xi_k\ e^{ik\frac{q2\pi}{2M}} \right\} \qquad (V.24)$$

Similarly, the Fourier coefficients of $\xi(\vartheta)$ are

$$(\xi)_q = \frac{1}{2\pi} \int_0^{2\pi} \xi(\vartheta)\ e^{iq\vartheta} = \frac{1}{2\pi} \sum_{k=1}^{2M} \int_{\vartheta_k - \frac{\eta}{2}}^{\vartheta_k + \frac{\eta}{2}} \left[\xi(\vartheta_k) + \frac{\vartheta - \vartheta_k}{\eta}\ \Delta\xi_k \right] e^{iq\vartheta}\, d\vartheta$$

$$= \frac{e^{-\frac{iq\pi}{2M}}}{\pi q} \left\{ \left[\sin\frac{q\eta}{2}\right] \sum_{k=1}^{2M} \xi(\vartheta_k)\ e^{ikq\frac{2\pi}{2M}} + \right.$$

$$\left. \frac{i}{2}\left[\frac{\sin\frac{q\eta}{2}}{\frac{q\eta}{2}} - \cos\frac{q\eta}{2} \right] \sum_{k=1}^{2M} \Delta\xi_k\ e^{ikq\frac{2\pi}{2M}} \right\} \qquad (V.25)$$

If $\frac{q}{M} \ll 1$ (the validity of the perturbation Hamiltonian (V.21) is restricted to harmonic orders much smaller than M), (V.24) and (V.25) become approximately

$$(n\xi)_q = \frac{\hat{n}e^{-\frac{iq\pi}{2M}}}{2M}\left\{\frac{i}{2}\left(\frac{r_o}{r_m}\right)^2\frac{q\pi}{2M}\sum_k\left[(-1)^k\,\xi(\vartheta_k)e^{ikq\frac{2\pi}{2M}}\right]+\frac{1}{4}\,\frac{r_o}{r_m}\sum_k\left[(-1)^k\Delta\xi_k e^{ikq\frac{2\pi}{2M}}\right]\right\} \tag{V.24a}$$

$$(\xi)_q = \frac{e^{-\frac{iq\pi}{2M}}}{2M}\left\{\frac{r_o}{r_m}\sum_q \xi(\vartheta_k)e^{ikq\frac{2\pi}{2M}} + \frac{i}{6}\left(\frac{r_o}{r_m}\right)^2\frac{q\pi}{2M}\sum_k \Delta\xi_k e^{ikq\frac{2\pi}{2M}}\right\} \tag{V.25a}$$

The complete forcing terms in (V.22) (V.23) may now be written in the form of a Fourier series

$$Q^2\xi - \left(\frac{r_m}{r_o}\right)^2\left[n\xi + r_o\,\frac{\delta B_z}{B_o}\right] = Q^2\sum_q \Xi_q\, e^{-iq\vartheta} \tag{V.26}$$

$$Q^2\zeta + \left(\frac{r_m}{r_o}\right)^2\left[n\zeta - r_o\varepsilon\right] = Q^2\sum_q Z_q\, e^{-iq\vartheta} \tag{V.27}$$

with Fourier amplitudes (using the approximations (V.24a) (V.25a))

$$\Xi_q = \frac{e^{-i\frac{q\pi}{2M}}}{2M}\left\{\frac{r_o}{r_m}\sum_{k=1}^{2M}\left[\xi(\vartheta_k) - \left(\frac{r_m}{r_o}\right)^2\frac{\hat{n}}{4Q^2}(-1)^k\,\Delta\xi_k\right]e^{ikq\frac{2\pi}{2M}}\right.$$

$$\left. + \left(\frac{r_o}{r_m}\right)^2\frac{i}{2}\,\frac{q\pi}{2M}\sum_{k=1}^{2M}\left[\frac{1}{3}\Delta\xi_k - \left(\frac{r_m}{r_o}\right)^2\frac{\hat{n}}{Q^2}(-1)^k\,\xi(\vartheta_k)\right]e^{ikq\frac{2\pi}{2M}}\right\} - \left(\frac{r_m}{r_o}\right)^2\frac{r_o}{Q^2}\frac{\delta B_q}{B_o} \tag{V.28}$$

$$Z_q = \frac{e^{-i\frac{q\pi}{2M}}}{2M}\left\{\frac{r_o}{r_m}\sum_{k=1}^{2M}\left[\zeta(\vartheta_k) + \left(\frac{r_m}{r_o}\right)^2\frac{\hat{n}}{4Q^2}(-1)^k\,\Delta\zeta_k\right]e^{ikq\frac{2\pi}{2M}}\right.$$

$$\left. + \left(\frac{r_o}{r_m}\right)^2\frac{i}{2}\,\frac{q\pi}{2M}\sum_{k=1}^{2M}\left[\frac{1}{3}\Delta\zeta_k + \left(\frac{r_m}{r_o}\right)^2\frac{\hat{n}}{Q^2}(-1)^k\,\zeta(\vartheta_k)\right]e^{ikq\frac{2\pi}{2M}}\right\} - \left(\frac{r_m}{r_o}\right)^2\frac{r_o}{Q^2}\varepsilon_q \tag{V.29}$$

If these Fourier coefficients of the forcing terms are known, it is easy to calculate the distorted closed orbit in Fourier series representation in the absence of non-linearities. Disregarding also the linear coupling term $2\epsilon n\bar{z}$, the radial closed orbit deviation resulting from (V.22) and (V.26) is

$$\bar{c}(\vartheta) = \sum_{q=-\infty}^{+\infty} c_q \, e^{-iq\vartheta} = \sum_q \frac{Q^2}{Q^2 - q^2} \Xi_q \, e^{-iq\vartheta} \qquad (V.30)$$

From this, the root mean square deviation of $\bar{c}$ follows as

$$< \bar{c}^2 >^{\frac{1}{2}} = \left(\sum_q |c_q|^2 \right)^{\frac{1}{2}} = \left(\sum_q \left| \frac{Q^2}{Q^2 - q^2} \Xi_q \right|^2 \right)^{\frac{1}{2}} \qquad (V.31)$$

A similar formula holds for the vertical deviation. For statistically independent $\xi(\vartheta_k)$ *and* $\Delta\xi_k$ *, or* $\zeta(\vartheta_k)$ *and* $\Delta\zeta_k$ *resp., expectation values* of $|\Xi_q|^2$ and $|Z_q|^2$ can be calculated from (V.28) (V.29) :

$$<|\Xi_q|^2> = \frac{1}{2M}\left\{ \left(\frac{r_m}{Q^2}\right)^2 < \left(\frac{\delta B}{B_o}\right)^2 > + \left[\left(\frac{r_o}{r_m}\right)^2 + \left(\frac{q\pi}{2M}\,\frac{\hat{n}}{2Q^2}\right)^2 \right] <\xi^2> + \left[\left(\frac{r_m}{r_o}\,\frac{\hat{n}}{4Q^2}\right)^2 + \left(\frac{1}{6}\,\frac{r_o^2}{r_m^2}\,\frac{q\pi}{2M}\right)^2 \right] <\Delta\xi^2> \right\} \qquad (V.32)$$

$$<|Z_q|^2> = \frac{1}{2M}\left\{ \left(\frac{r_m}{Q^2}\right)^2 <\epsilon^2> + \left[\left(\frac{r_o}{r_m}\right)^2 + \left(\frac{q\pi}{2M}\,\frac{\hat{n}}{2Q^2}\right)^2 \right] <\zeta^2> + \left[\left(\frac{r_m}{r_o}\,\frac{\hat{n}}{4Q^2}\right)^2 + \left(\frac{1}{6}\,\frac{r_o^2}{r_m^2}\,\frac{q\pi}{2M}\right)^2 \right] <\Delta\zeta^2> \right\}$$

or, numerically for CERN PS figures,

$$<|\Xi_q|^2> = \frac{1}{100}\left\{ 6.6\cdot 10^4 < \left(\frac{\delta B}{B_o}\right)^2 > + [0.49 + 0.0130\, q^2] <\xi^2> + [6.7 + 6.3\cdot 10^{-6} q^2] <\Delta\xi^2> \right\} \qquad (V.32a)$$

On the right hand side, the expectation values of $\left(\frac{\delta B}{B_o}\right)^2$, ξ^2, $\Delta\xi^2$, ϵ^2, ζ^2 and $\Delta\zeta^2$ *appear*.

The numerical coefficients show to what degree the $(\frac{1}{2}\ F\ \frac{1}{2}\ D)$ combination reduces the effect of unit translations ξ, ζ with respect to that of tilts $\Delta\xi$, $\Delta\zeta$. If we assume $<\Delta\xi^2>$ and $<\xi^2>$ of equal order of magnitude, $<|\Xi_q|^2>$ is almost independent of q (for moderate q; it should be remembered that the validity of our approximations is restricted to q much smaller than M). Under this assumption a rough estimate of the r.m.s. closed orbit deviation can be made by (V.31) :

$$\sqrt{< c^2 >} = \sqrt{< |\Xi|^2 >} \sqrt{\sum \left| \frac{Q^2}{Q^2 - q^2} \right|^2} \qquad (V.33)$$

(V.33) together with (V.32) determines the admissible tolerances for alignment and guide field errors by fixing an upper limit for the closed orbit deviations. Actually, the limit is fixed for the maximum closed orbit deviation rather than for the r.m.s. deviation. The maximum deviation, to which the closed orbit harmonics can add up is

$$\left|\bar{c}_{max}\right| \leq \sum_q \left|\frac{Q^2}{Q^2 - q^2}\right| \left|\Xi_q\right| \cong \left|\Xi\right| \sum_q \left|\frac{Q^2}{Q^2 - q^2}\right|, \qquad (V.34)$$

assuming again $\left|\Xi_q\right|$ independent of q.

To show how much $\bar{c}_{max}$ may differ from $\sqrt{\langle c^2 \rangle}$, the sums appearing in (V.33) and (V.34) (added from q = - 12 to + 12) are given in the table below for 3 values of Q in the neighbourhood of the working point of the CERN PS.

		Q = 6.10	6.25	6.40
$\sqrt{\sum_{q=-12}^{+12}{}' \left(\frac{Q^2}{Q^2 - q^2}\right)^2}$	=	44.2	19.9	15.0
$\sum_{-12}^{+12}{}' \left|\frac{Q^2}{Q^2 - q^2}\right|$	=	92.6	58.4	51.8
$\sqrt{2}\,\frac{Q^2}{Q^2 - 6^2}$	=	43.5	18.0	11.7
$2\,\frac{Q^2}{Q^2 - 6^2}$	=	61.5	25.5	16.5

The closed orbit deviation can add up to 2 to 3 times its r.m.s. value by interference. The terms for which $|q|$ is closest to Q make predominant contributions to the sums. The r.m.s. value and the amplitude of these terms are given in the bottom lines of the table, in order to bring this into evidence.

To obtain figures for <u>tolerable guide field and misalignment errors</u>, statistical independence of the positions of magnet unit ends may be assumed (for simplicity and lack of better knowledge). Then $\langle \Delta\xi^2 \rangle = 2 \langle \xi^2 \rangle$, and from (V.32a)

$$\langle \Xi^2 \rangle = 6.6.10^2 \langle (\frac{\delta B}{B_0})^2 \rangle + 13.4.10^{-2} \langle \xi^2 \rangle \quad [cm^2]$$

(smaller terms omitted). Distributing the tolerance equally between $\frac{\delta B}{B_0}$ and ξ we find as tolerable limits

$$\sqrt{<(\frac{\delta B}{B_0})^2>} = 0.027\sqrt{<\Xi^2>} = \frac{0.027}{60}\,|c_{max}\,[cm]| \approx 5.10^{-4}|c_{max}|$$

$$\sqrt{<\xi^2>} = 1.9\sqrt{<\Xi^2>} = \frac{1.9}{60}\,|c_{max}| \approx \frac{|c_{max}|}{30}$$

taking for $\frac{|c_{max}|}{\sqrt{<\Xi^2>}}$ the figures of the above table at $Q = 6.25$.

Thus a tolerable maximum deviation of the closed orbit of $|c_{max}| = 2$ cm would result in admissible r.m.s. errors of 10^{-3} for $\frac{\delta B}{B_0}$ and 0.06 cm for ξ.

More careful estimates of the tolerable errors had been based on the probability of the closed orbit exceeding a given limit $|c_{max}|$, certain statistical properties of the errors assumed (Adams and Hine [1953d], Lüders [1953b, 1955]). Practically the same tolerances were found as in the rough estimate given here.

It should be remarked that the tolerances given for alignment errors do not necessarily apply to the precision of the circular shape of the ring magnet in the large. Smooth, systematic misalignments (e.g. caused by systematic errors in the setting-up procedure, or by subsequent deformation of the foundation) are followed by the closed orbit. For example, consider a sinusoidal misalignment

$$\xi = \xi_0 \cos q\vartheta, \quad q << Q ,$$

then the forcing term in equation (V.22) becomes

$$Q^2\xi(\vartheta)-(\frac{r_m}{r_o})^2 n(\vartheta)\xi(\vartheta) = Q^2\xi_0\cos q\vartheta-(\frac{r_m}{r_o})^2\xi_0[n_M\cos(M+q)\vartheta+n_M\cos(M-q)\vartheta + ..]$$

where, for $q << Q$, the first term determines the major part of the closed orbit deviation :

$$\bar{c}(\vartheta) = \frac{Q^2}{Q^2-q^2}\,\xi_0\cos q\vartheta-(\frac{r_m}{r_o})^2\xi_0\left[n_M\frac{\cos(M+q)\vartheta}{Q^2-(M+q)^2} + n_M\frac{\cos(M-q)\vartheta}{Q^2-(M-q)^2} + \dots\right.$$

$$\approx \xi(\vartheta)-(\frac{r_m}{r_o})^2\xi_0\left[n_M\frac{\cos(M+q)\vartheta}{Q^2-(M+q)^2} + n_M\frac{\cos(M-q)\vartheta}{Q^2-(M-q)^2} + ..\right]$$

The second part on the r.h.s. happens to be very nearly equal to $-\xi(\vartheta)(\rho_1 \cos M\vartheta + \rho_3 \cos 3M\vartheta + ..)$, as $q \ll Q \ll M$ and approximately

$$\rho_\nu \approx -\frac{2n_{\nu M}}{(\nu M)^2}$$

from (IV.4). Thus

$$c(\vartheta) = \bar{c}(\vartheta)(1 + \rho_1 \cos M\vartheta + ..) = \xi(\vartheta)(1 - \rho_1 \cos M\vartheta - ..)(1 + \rho_1 \cos M\vartheta + ..)$$
$$\approx \xi(\vartheta) \; .$$

The same conclusion can be made plausible by going back to the original Hamiltonian (I.24, in non-smooth coordinates) and considering a uniform displacement ξ = const. Then the equation of motion can be written

$$\frac{d^2x}{d\vartheta^2} - n(\vartheta)(x - \xi) = 0,$$

showing that $c = \xi$ is the displaced closed orbit. This will still be approximately true for a very smoothly varying $\xi(\vartheta)$.

As the vacuum chamber will also follow smooth distortions of the ring magnet, they do not affect the position of the closed orbit with respect to the vacuum chamber in first approximation. The tolerances given before, therefore refer to random misalignments with respect to the smoothly distorted closed orbit.

The behaviour of the vertical closed orbit deviations is of course similar.

Whether the <u>influence of the linear coupling terms and the non-linearities</u> on the closed orbit is important or not can be estimated by comparing the corresponding forces with the alignment error forcing terms. It has been pointed out that, with rigid ($\frac{1}{2}$ F $\frac{1}{2}$ D) units, the coupling force is mainly due to twists $\Delta\epsilon$ of units. The force produced is therefore roughly $\hat{n}\,\Delta\,\epsilon\,\bar{z}$, and

$$\left|\frac{\hat{n}\,\Delta\,\epsilon\,\bar{z}}{Q^2\,\Xi}\right| \gtrsim 1 \quad \text{requires} \quad \Delta\,\epsilon \gtrsim \frac{Q^2}{\hat{n}}\left|\frac{\Xi}{\bar{z}}\right| \sim \frac{1}{7}\left|\frac{\Xi}{\bar{z}}\right|$$

Taking $\left|\frac{\Xi}{\bar{z}}\right| \sim \frac{1}{20}$

i.e. the order of magnitude following from (V.34) and the table on page 121 twists $\Delta\epsilon \gtrsim \frac{1}{140}$ would be required to influence the closed orbit distortion strongly.

For the non-linear forces we have

$$\left|\frac{\frac{1}{2}\frac{dn}{dx}\bar{x}^2}{Q^2\,\Xi}\right| \gtrsim 1 \quad \text{if} \quad \frac{dn}{dx}\bar{x} = \Delta n_{\text{quadratic}} \gtrsim 2Q^2\left|\frac{\Xi}{\bar{x}}\right| \sim 4$$

$$\left|\frac{\frac{1}{3!}\frac{d^2n}{dx^2}\bar{x}^3}{Q^2\,\Xi}\right| \gtrsim 1 \quad \text{if} \quad \frac{1}{2}\frac{d^2n}{dx^2}\bar{x}^2 = \Delta n_{\text{cubic}} \gtrsim 3Q^2\left|\frac{\Xi}{\bar{x}}\right| \sim 6$$

So non-linear forces do affect the closed orbit if the relative change $\frac{\Delta n}{n}$ by closed orbit deviations is more than about one percent ($\hat{n}$= 282 in our numerical examples). In order to obtain an at least qualitative picture of this influence, we assume that only the harmonic closest to Q is present in the forcing term. This simplification is not too far from reality according to a previous paragraph. If we furthermore confine ourselves to only radial perturbations, the equation of motion (V.22) takes the form

$$\bar{x}'' + Q^2\bar{x} = \frac{p_0}{p}\left\{r_m\frac{\delta p}{p_0} + 2Q^2\,\Xi\cos q\vartheta + 2Q^2\frac{\delta p}{p_0}\bar{x} + \frac{1}{2}\left(\frac{r_m}{r_0}\right)^2\left(\frac{dn}{dx}\right)_0\bar{x}^2\right.$$
$$\left. + \frac{1}{3!}\left(\frac{r_m}{r_0}\right)^2\left[\left(\frac{d^2n}{dx^2}\right)_0 + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M + \ldots\right]\bar{x}^3\right\}$$

$$\bar{z} = 0$$

The closed orbit solution contains, except terms of frequencies zero and q, also multiples of q as frequencies because of the non-linear terms. The higher harmonics being small we may, however, consider an approximate solution

$$\bar{x} = \bar{c}(\vartheta) = c_0 + \tilde{c}\cos q\vartheta \tag{V.36}$$

It is of the same form as in the linear case, but mean displacement c_0 and amplitude $\tilde{c}$ will now be affected by the non-linearities. By introducing (V.36) and

$$\bar{x}^2 = c_0^2 + 2c_0 \tilde{c} \cos q\vartheta + \frac{\tilde{c}^2}{2} (1 + \cos 2q\vartheta)$$

$$\bar{x}^3 = c_0^3 + \frac{3}{2} c_0 \tilde{c}^2 + 3\tilde{c} (c_0^2 + \frac{\tilde{c}^2}{4}) \cos q\vartheta + \ldots$$

into (V.35). equating the constant and $\cos q\vartheta$ terms singly, and neglecting higher harmonics, two equations determing c_0 and $\tilde{c}$ as a function of $\frac{\delta p}{p_0}$, Ξ, and Q result :

$$Q^2 c_0 = \frac{p_0}{p} \left\{ (r_m + 2Q^2 c_0) \frac{\delta p}{p_0} + \frac{n_x}{2} (c_0^2 + \frac{\tilde{c}^2}{2}) + \frac{n_{xx}}{3!} (c_0^3 + \frac{3}{2} c_0 \tilde{c}^2) \right\}$$

$$(Q^2 - q^2)\tilde{c} = \frac{p_0}{p} \left\{ 2Q^2 \Xi + 2Q^2 \frac{\delta p}{p_0} \tilde{c} + n_x c_0 \tilde{c} + \frac{n_{xx}}{2} \tilde{c} (c_0^2 + \frac{\tilde{c}^2}{4}) \right\} . \qquad (V.37)$$

Here n_x and n_{xx} have been written as abbreviations for the coefficients of the non-linear terms in (V.35) :

$$n_x \equiv (\frac{r_m}{r_0})^2 (\frac{dn}{dx})_0 \; ; \; n_{xx} \equiv (\frac{r_m}{r_0})^2 \left[(\frac{d^2 n}{dx^2})_0 + 4\rho_1 (\frac{d^2 n}{dx^2})_M + \ldots \right] .$$

With a few minor neglections, we can write the equations (V.37) in the following form convenient for graphical solution

$$\frac{2\Xi}{\tilde{c}} + \frac{n_{xx}}{2Q^2} \frac{\tilde{c}^2}{4} = \frac{1}{Q^2} \left[Q^2 - q^2 - n_x c_0 - \frac{n_{xx}}{2} c_0^2 - 2Q^2 \frac{\delta p}{p_0} \right] \qquad (V.38)$$

$$\frac{\delta p}{p_0} = \frac{1}{r_m} \left[Q^2 c_0 - (\frac{n_x}{2} + \frac{Q^4}{r_m}) c_0^2 - \frac{n_{xx}}{3!} c_0^3 - \frac{1}{2} (\frac{n_x}{2} + \frac{n_{xx}}{2} c_0)\tilde{c}^2 \right] \qquad (V.39)$$

In the diagrams (fig. 17-22), the left and right hand side of (V.38) are plotted horizontally as functions of respectively $\tilde{c}$, c_0. In plotting the r.h.s., (V.39) has been used, and the dependence of $\frac{\delta p}{p_0}$ on $\tilde{c}$ has been accounted for by iterative correction. Physically, the horizontal coordinate in the diagrams is the relative difference of the square of the "effective" frequency and q^2, the effective frequency $(Q + \delta Q_1)$ resulting from the modification of the unperturbed Q-value by deviations in momentum and radius. The l.h.s. curve $\tilde{c}$ represents the (non-linear) resonance curve of the forced oscillation.

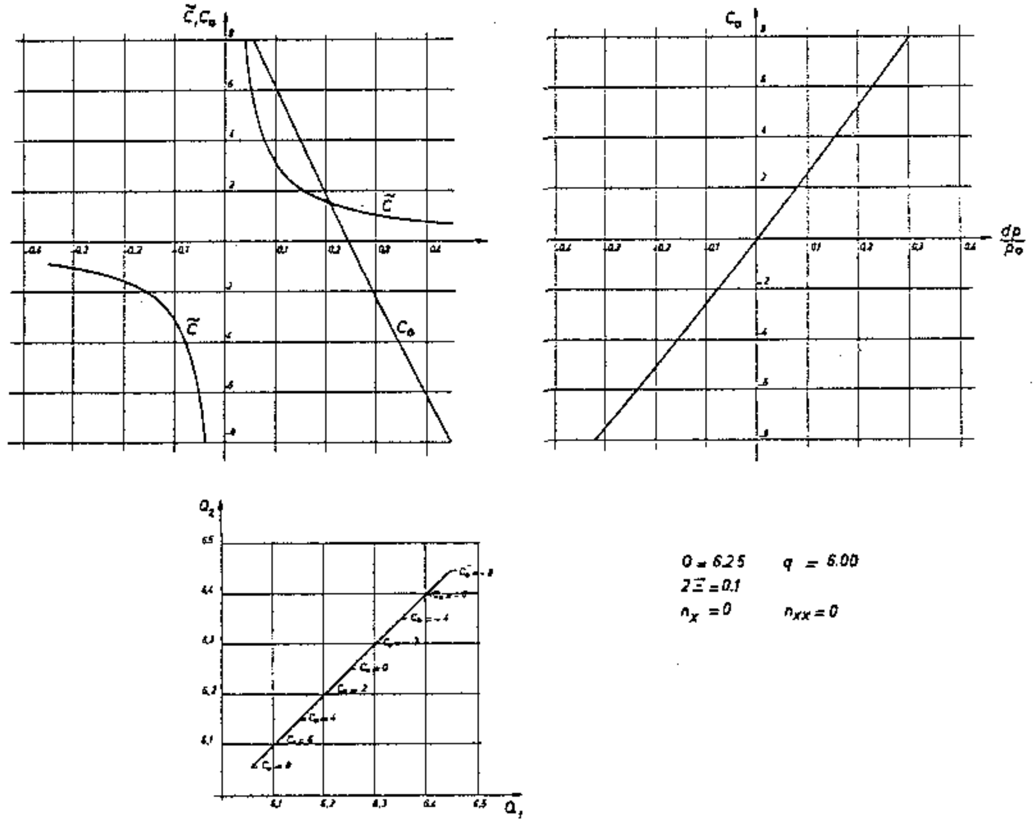

Fig. 17 Interrelations between displacement c_o of closed orbit and amplitude $\tilde{c}$ of closed orbit, momentum deviation $\frac{\delta p}{p_o}$, and shift of Q_1 and Q_2. Ideal linear field.

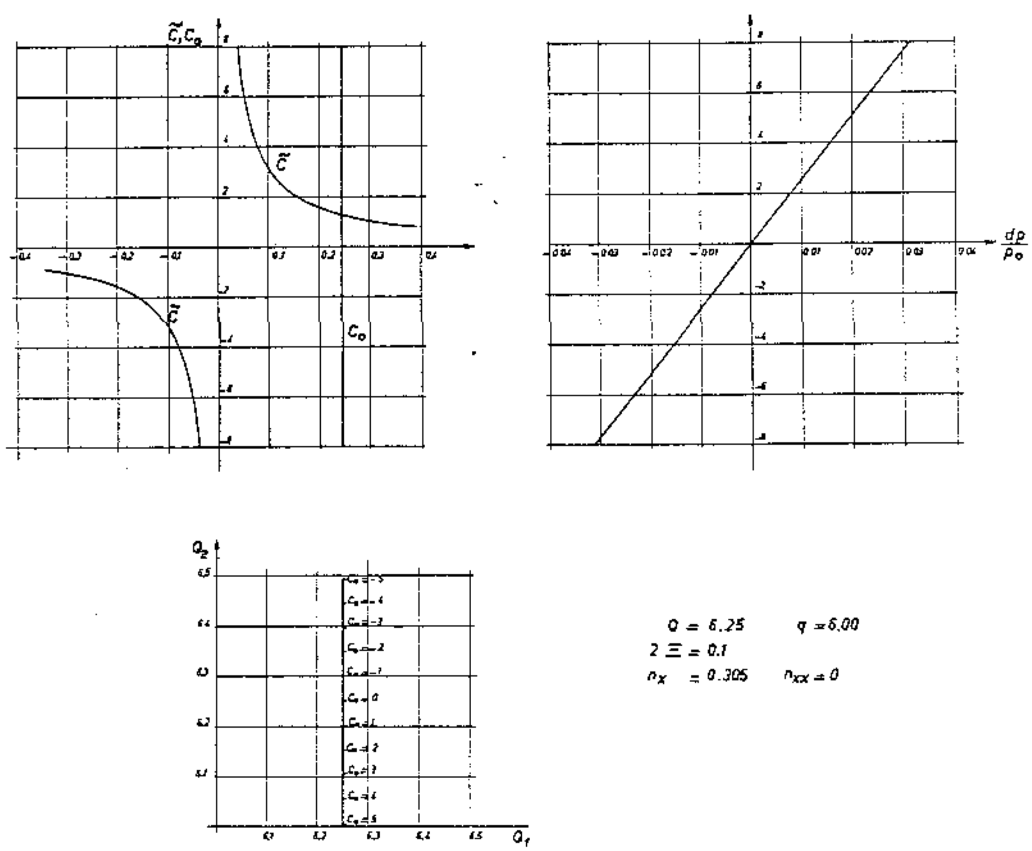

Fig. 18 Interrelations between displacement c_o of closed orbit and amplitude $\tilde{c}$ of closed orbit, momentum deviation $\frac{\delta p}{p_o}$, and shift of Q_1 and Q_2. With quadratic non-linearity to suppress dependence of Q_1 on momentum.

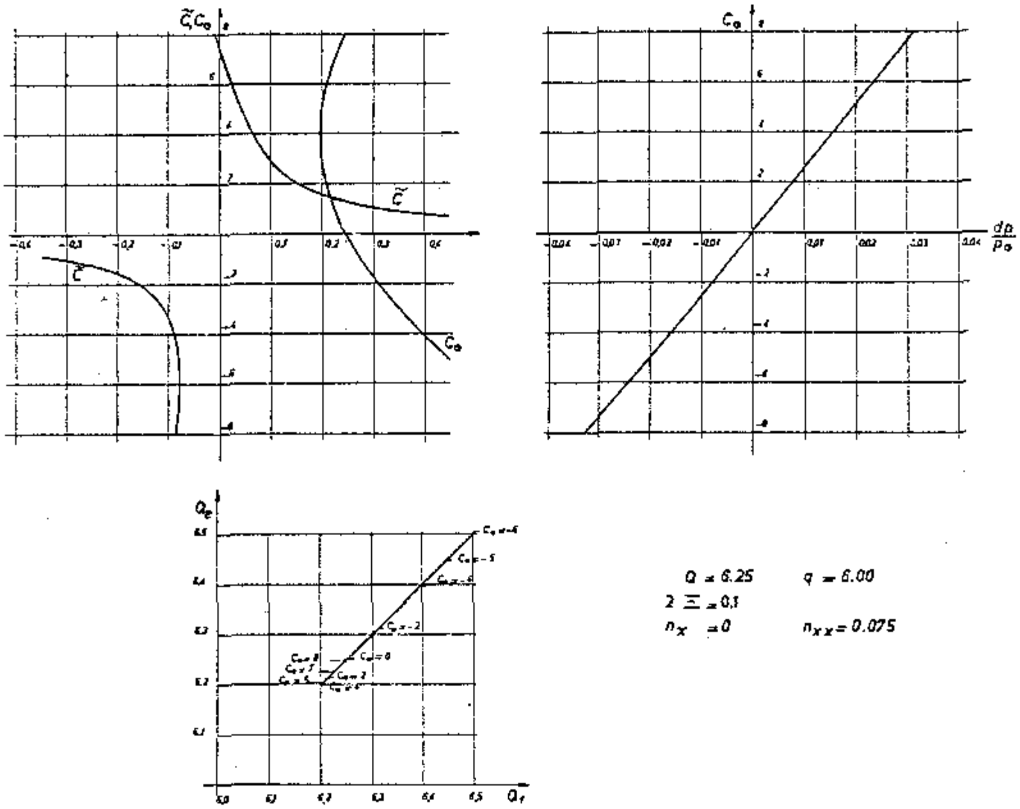

Fig. 19 Interrelations between displacement c_0 of closed orbit and amplitude $\tilde{c}$ of closed orbit, momentum deviation $\frac{\delta p}{p_0}$, and shift of Q_1 and Q_2. With cubic non-linearity to repell Q_1 and Q_2 simultaneously from integral value.

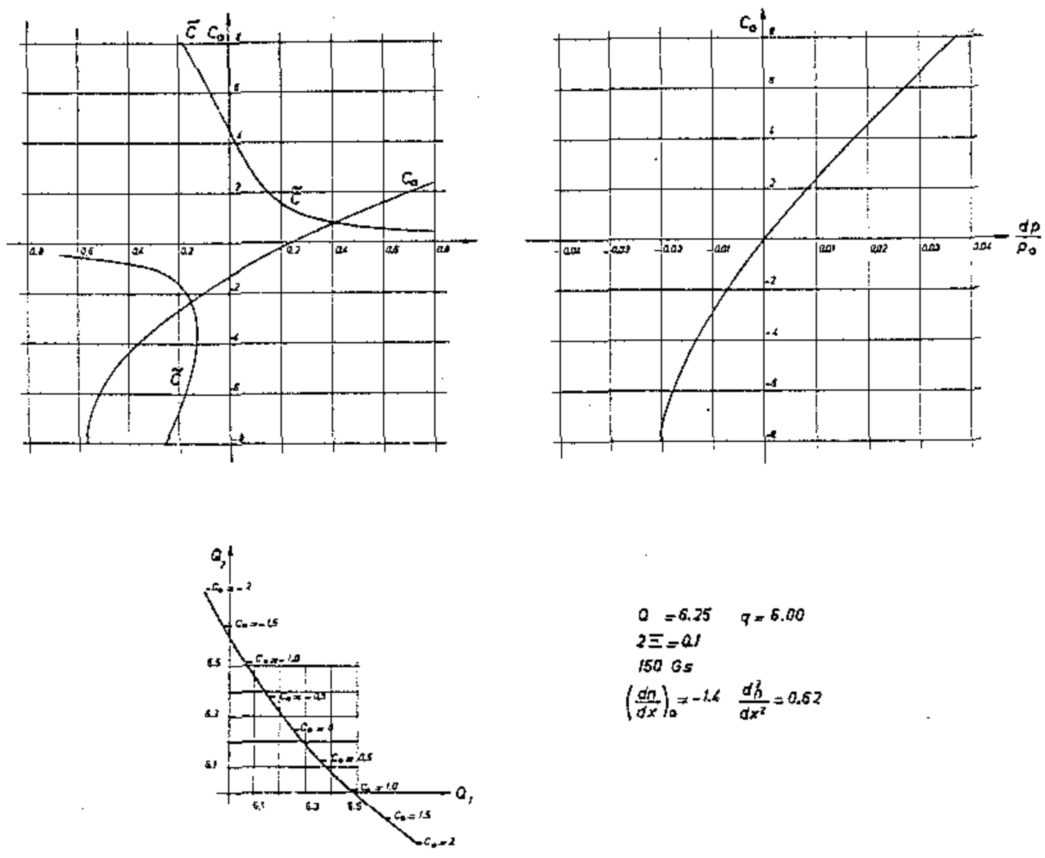

Fig. 20 Interrelations between displacement c_0 of closed orbit and amplitude $\tilde{c}$ of closed orbit, momentum deviation $\frac{\delta p}{p_0}$, and shift of Q_1 and Q_2. For CERN PS non-linearities not corrected, Q-values for $c_0 = 0$ kept adjusted to nominal value 6.25. Range of displacement restricted to $c_0 = \pm 2$ cm, as field is represented fairly accurately by quadratic and cubic non-linearities within this range.

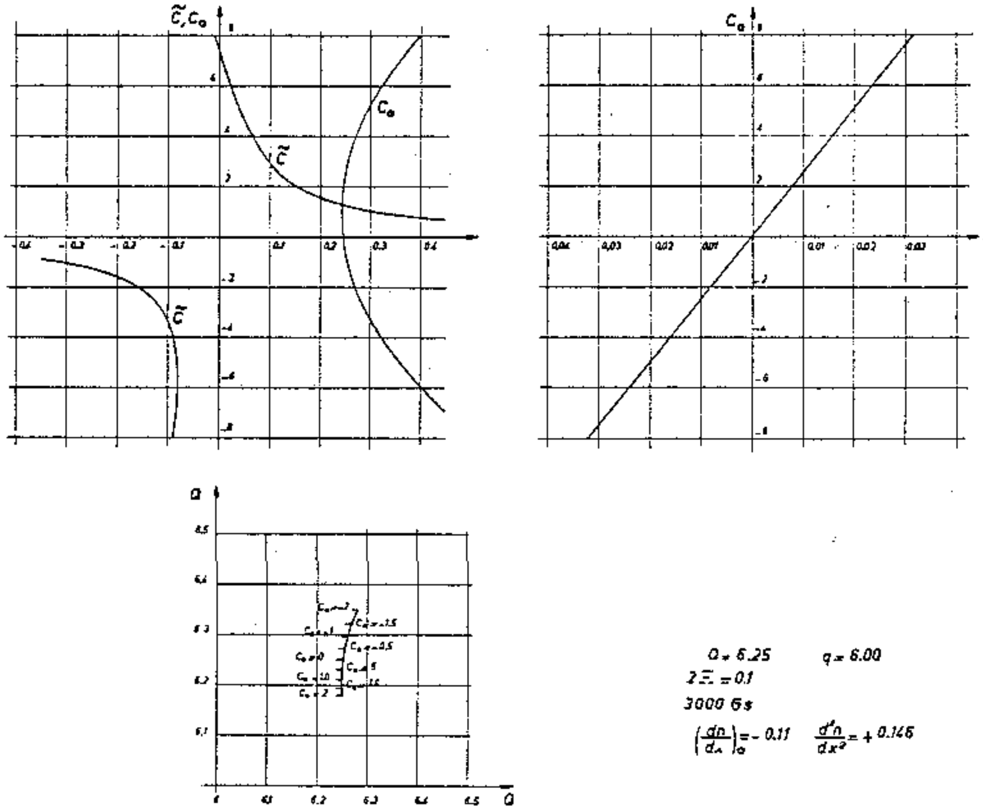

Fig. 21 Interrelations between displacement c_0 of closed orbit and amplitude $\tilde{c}$ of closed orbit, momentum deviation $\frac{\delta p}{p_0}$, and shift of Q_1 and Q_2. For CERN PS non-linearities not corrected, Q-values for $c_0 = 0$ kept adjusted to nominal value 6.25. Range of displacement restricted to $c_0 = \pm 2$ cm, as field is represented fairly accurately by quadratic and cubic non-linearities within this range.

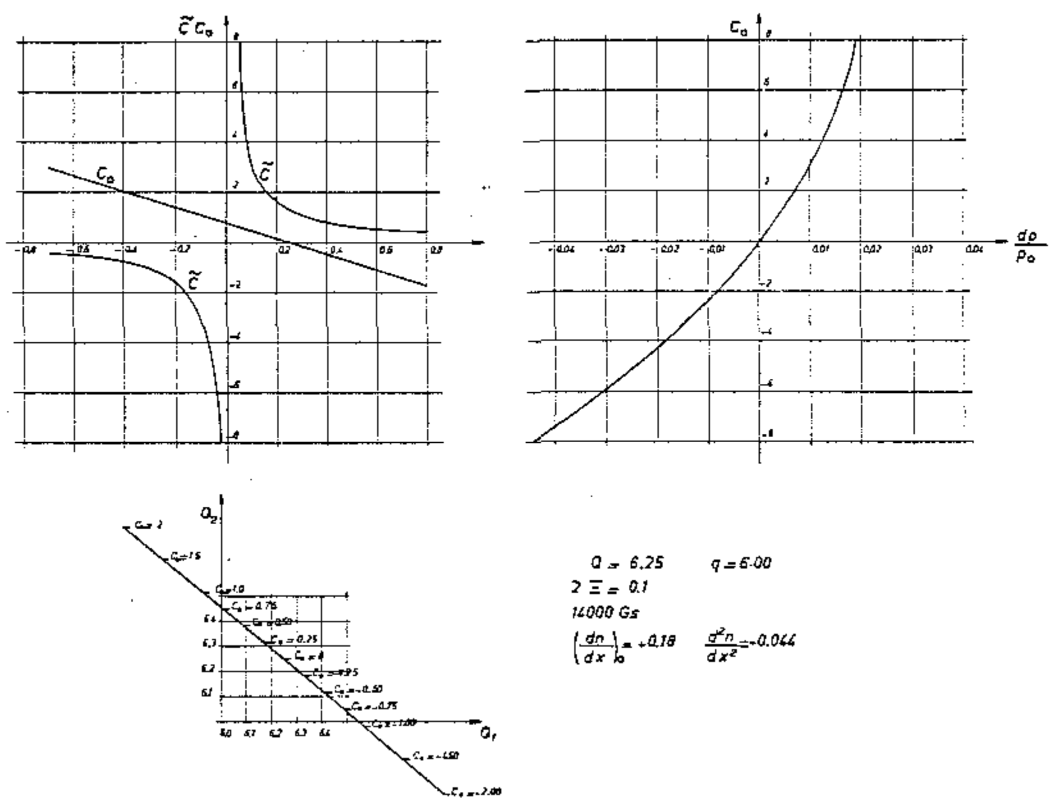

Fig. 22 Interrelations between displacement c_0 of closed orbit and amplitude $\tilde{c}$ of closed orbit, momentum deviation $\frac{\delta p}{p_0}$, and shift of Q_1 and Q_2. For CERN PS non-linearities not corrected, Q-values for $c_0 = 0$ kept adjusted to nominal value 6.25. Range of displacement restricted to $c_0 = \pm 2$ cm, as field is represented fairly

The diagrams are used by first finding the effective frequency to a given mean displacement c_0 of the closed orbit, and then reading off at the same frequency (abscissa) the amplitude $\tilde{c}$ of the closed orbit oscillation. Separate diagrams show the relation of the mean displacement c_0 to the momentum deviation $\frac{\delta p}{p_0}$.

The diagrams given represent data of the CERN P.S. first for a perfectly linear field, then for a set of typical non-linearities, as will be encountered without any correction devices applied, and finally for non-linearities corrected by sextupole and octupole lenses in such a way that either $\delta Q_1 \approx 0$ independent of momentum, or δQ_1, δQ_2 always move away from the integral resonances for any closed orbit displacement, in order to minimize the closed orbit amplitude.

The sextupoles and octupoles required can be derived from the formulae for the small oscillation frequency shifts (App. VI) (these formulae can also be inferred easily from (V.22,23)) :

$$\delta Q_1 = - Q\,\frac{\delta p}{p_0} - \left(\frac{r_m}{r_0}\right)^2 \left[\frac{1}{2Q}\left(\frac{dn}{dx}\right)_0 c_0 + \frac{1}{4Q}\left(\left(\frac{d^2n}{dx^2}\right)_0 + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M \cdot\cdot\right) c_0^2\right]$$

$$\delta Q_2 = - Q\,\frac{\delta p}{p_0} + \left(\frac{r_m}{r_0}\right)^2 \left[\frac{1}{2Q}\left(\frac{dn}{dx}\right)_0 c_0 + \frac{1}{4Q}\left(\left(\frac{d^2n}{dx^2}\right)_0 c_0^2\right]\right. .$$

With regard to the case considered, the vertical closed orbit displacement has been assumed as zero; in addition a few minor approximations have been made here as $\frac{p_0}{p} \approx 1$, $w_0^2 \approx \frac{1}{2Q}$, $\rho_2 \approx 0$, which are meaningless for the present purpose. Introducing $\frac{\delta p}{p_0}$ from (V.39) and neglecting again small terms results in

$$\delta Q_1 = -\left[\frac{Q^3}{r_m} + \frac{1}{2Q}\left(\frac{r_m}{r_0}\right)^2\left(\frac{dn}{dx}\right)_0\right] c_0 - \frac{1}{4Q}\left(\frac{r_m}{r_0}\right)^2 \left[\left(\frac{d^2n}{dx^2}\right)_0 + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M\right] c_0^2$$

$$\delta Q_2 = -\left[\frac{Q^3}{r_m} - \frac{1}{2Q}\left(\frac{r_m}{r_0}\right)^2\left(\frac{dn}{dx}\right)_0\right] c_0 + \frac{1}{4Q}\left(\frac{r_m}{r_0}\right)^2\left(\frac{d^2n}{dx^2}\right)_0 c_0^2 \ . \qquad \text{(V.40)}$$

The linear dependence of δQ_1 on c_0 is suppressed by making

$$\left(\frac{r_m}{r_0}\right)^2 \left(\frac{dn}{dx}\right)_0 = - \frac{2Q^4}{r_m} \approx - 0.30 \text{ cm}^{-1}.$$

$\left(\frac{dn}{dx}\right)_0$ is the average of the combined $\frac{dn}{dx}$ due to magnets and sextupole lenses. In terms of the magnet n-value, this non-linearity is very small (about $0.5\cdot 10^{-3}$ of n per cm). By suppressing the momentum dependence of δQ_1, δQ_2 becomes twice as sensitive to it as shown by (V.40). From the point of view of avoiding resonances, it is therefore more reasonable to make $\frac{dn}{dx} = 0$ and use octupole lenses to prevent δQ_1, δQ_2 from approaching the resonances at $\delta Q_1 = \delta Q_2 = -0.25$. Making for example

$$\frac{1}{4Q}\left(\frac{r_m}{r_o}\right)^2\left(\frac{d^2n}{dx^2}\right)_o = -\frac{1}{4Q}\left(\frac{r_m}{r_o}\right)^2\left[\left(\frac{d^2n}{dx^2}\right)_o + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M\right] = 0.0030 \text{ cm}^{-2},$$

i.e.

$$2\left(\frac{d^2n}{dx^2}\right)_o = -4\rho_1\left(\frac{d^2n}{dx^2}\right)_M = 0.075 \text{ cm}^{-2} \qquad \text{(V.41)}$$

the working point moves between $Q_1 = Q_2 = 6.45$ for $c_o = -5$ cm, over $Q_1 = Q_2 = 6.25$ at $c_o = 0$ and $Q_1 = Q_2 = 6.20$ at $c_o = +4$ cm to 6.20 at $c_o = +5$ cm. The non-linearities necessary are again very small in terms of the magnet n-value.

The requirement (V.41) can be met by arranging two sets OI and OII of octupole lenses an odd number m of half-periods apart. As shown in Appendix VI, they produce the harmonics

$$\left(\frac{d^2n}{dx^2}\right)^{\text{Octupoles}}_{\nu M} = \frac{S\pi}{2\pi}\ \sin\frac{\frac{\nu M\pi}{2}}{\frac{\nu M\pi}{2}}\left[\left(\frac{d^2n}{dx^2}\right)^{OI} + (-1)^{\nu m}\left(\frac{d^2n}{dx^2}\right)^{OII}\right].$$

So the even and odd harmonics are controlled by $\left(\frac{d^2n}{dx^2}\right)^{OI} + \left(\frac{d^2n}{dx^2}\right)^{OII}$ and $\left(\frac{d^2n}{dx^2}\right)^{OI} - \left(\frac{d^2n}{dx^2}\right)^{OII}$, respectively. The magnet octupole contribution has to be taken account of in setting the lenses for odd harmonics.

How the Q-values move with the displacement c_o of the closed orbit is shown separately in figs 17-22 by lines in the (Q_1, Q_2)-plane. These lines are modified somewhat by wavy distortions of the closed orbit because there is some influence of a wavy distortion on the relation between momentum and c_o (see (V.39)). Vertical distortions of the closed orbit would cause additional modifications of the movement of the Q-values. Their effect will in general, however, be smaller as nothing like the mean displacement due to momentum deviations is involved.

It is clear from the diagrams shown that, e.g. for a momentum range of $\pm 5.10^{-3}$ the closed orbits are kept within a reasonable central region of the vacuum chamber at medium fields, but that appropriate corrections of the non-linearities (apart from necessary corrections of the n-value on the equilibrium orbit) may be vital at very high, and perhaps also at very low fields. They are provided for by pole face windings plus sextupole and octupole lenses in the CERN PS. The sinusoidal forcing term assumed in the calculation of the diagrams ($2\Xi = 0.1$ cm) produces roughly the same maximum amplitude of the closed orbit as 0.05 cm r.m.s. random misalignments.

Actually there exists a <u>multiplicity of closed orbits</u> because of the non-linearity of the equations (V.22) or (V.35) resp. This is shown by the bent-over resonance curves in the diagrams fig.19-22. For the non-linearities tending towards zero, one of the closed orbits goes over into $x = 0$ whereas the others disappear towards infinity. (In most of the cases represented by figs.19-22 they would be outside the vacuum chamber region). The question arising is : Which of these closed orbits are equilibrium orbits for free betatrons oscillations?

Representing the particle motion by its phase space path (taking as phase, as in section 9, the phase of the particle oscillation with respect to the phase of the perturbation which causes the closed orbit distortion), the closed orbits are the fixed points in the phase path pattern. Small oscillations are possible about stable fixed points, whereas large excursions can take place about instable fixed points. The topology of the phase space pattern may be studied on equation (V.35), simplified by retaining only the essential perturbation terms :

$$\bar{x}'' + Q^2 \bar{x} = 2\Xi Q^2 \cos p\vartheta + \frac{n_{xx}}{3!}\bar{x}^3, \quad p = \text{integer.} \tag{V.42}$$

Following (V.36) to (V.38), the amplitude of the closed orbit

$$\bar{c}(\vartheta) = \tilde{c} \cos p\,\vartheta$$

is given by

$$\frac{2\Xi Q^2}{\tilde{c}} + \frac{n_{xx}}{8}\tilde{c}^2 = Q^2 - p^2 \tag{V.42a}$$

and represented graphically in fig. 23 for numerical values close to those of the preceding diagrams.

The phase paths are conveniently calculated by means of the invariant (9.6) (for first order resonance) :

$$C = (Q - p)\,A + \overline{\mathcal{H}}^{(1)} = (Q - p + h_{110})\,A + h_{220}\,A^2 + h_{20,2p}\,A\,e^{i2[(Q-p)\vartheta + \phi]} + h_{21p}\,A^{3/2}\,e^{i[(Q-p)\vartheta + \phi]} + \text{conj. complex.} \quad (V.43)$$

$\overline{\mathcal{H}}^{(1)}$ is the "low frequency" part of the perturbation Hamiltonian in terms of coordinates with reference to (any) one of the closed orbits. In this way, terms linear in the coordinates or $A^{1/2}$ are suppressed in the Hamiltonian, which was shown in Section 8 to be necessary for the low frequency approximation to hold generally. For the present case (V.42), the perturbation Hamiltonian

$$H^{(1)} = V_1 x + V_4 x^4 = -(2\,\Xi\,Q^2 \cos p\vartheta)x - \frac{n_{xx}}{4!}x^4 \quad (V.44)^1$$

becomes in terms of the coordinate $y = x - c(\vartheta)$ relative to a closed orbit, according to (VI.7)

$$\mathcal{H}^{(1)} = 6V_4c^2y^2 + 4V_4cy^3 + V_4y^4 = -n_{xx}\left[\frac{\tilde{c}^2}{8}(1+\cos 2p\vartheta)y^2 + \frac{\tilde{c}}{6}(\cos p\vartheta)y^3 + \frac{1}{24}y^4\right]$$

Using this and Section 6 or Appendix VI, we obtain the coefficients

$$h_{110} = v_{110} = -2w_0^2\,\frac{n_{xx}\tilde{c}^2}{8} = -\frac{n_{xx}\tilde{c}^2}{8Q}$$

$$h_{220} = v_{220} = -6w_0^4\,\frac{n_{xx}}{24} = -\frac{n_{xx}}{16Q^2}$$

$$h_{20,2p} = v_{20,2p} = -w_0^2\,\frac{n_{xx}\tilde{c}^2}{16} = -\frac{n_{xx}\tilde{c}^2}{32Q}$$

$$h_{31p} = v_{21,p} = -3w_0^3\,\frac{n_{xx}\tilde{c}}{12} = -\frac{n_{xx}\tilde{c}}{8Q\sqrt{2Q}}$$

(in first approximation which is good enough here). The invariant (V.43) can now be written

$$C = \left[Q - p - \frac{n_{xx}\tilde{c}^2}{8Q}\left(1 + \frac{1}{2}\cos 2\psi\right)\right]A - \frac{n_{xx}\tilde{c}}{4\sqrt{2}\,Q^{3/2}}A^{3/2}\cos\psi - \frac{n_{xx}}{16Q^2}A^2$$

$$= \frac{R^2}{4}\left[2Q(Q-p) - \frac{n_{xx}\tilde{c}^2}{8} - \frac{n_{xx}}{4}\left(\frac{R}{2} + \tilde{c}\cos\psi\right)^2\right] \quad (V.45)$$

where

$$\psi = (Q - p)\,\vartheta + \phi$$

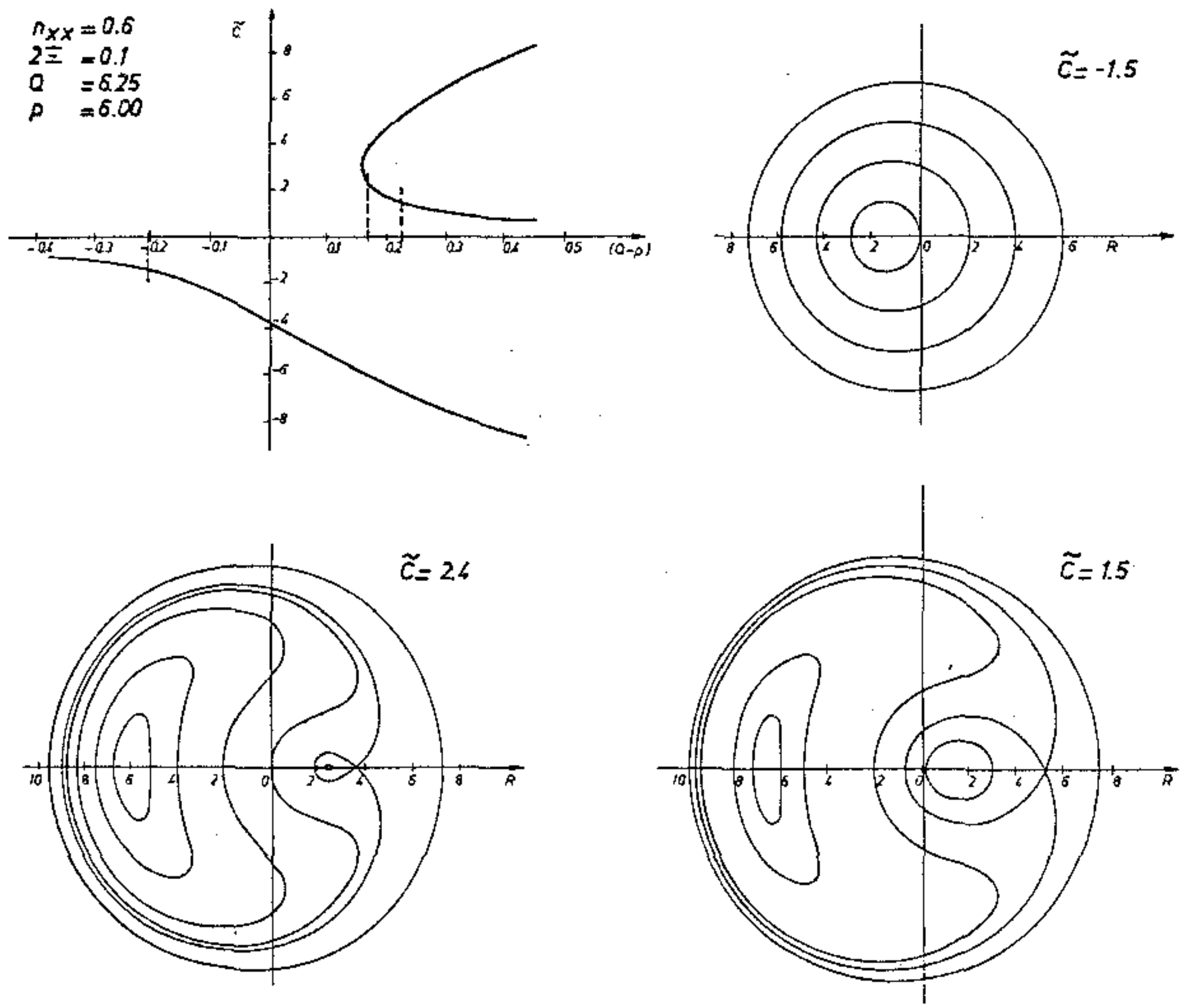

Fig. 23 Non-linear resonance curve and phase diagrams of non-linear betatron oscillations showing multiple closed orbits as fixed points.

and the actual amplitude

$$R = \left(\frac{2A}{Q}\right)^{\frac{1}{2}}$$

has been introduced.

It may be noted, here that in many cases (in particular in the present example) the invariant derived directly from the low frequency part of the Hamiltonian (V.44) does describe the motion in good approximation. It has been shown in Section 8 that linear terms in the Hamiltonian can make the rapidly varying part of the solution too large for the "low frequency" approximation to hold. This argument becomes effective only if small enough amplitudes occur in the motion. If this is not the case (or if zero amplitudes are passed fast, the use of coordinates with reference to a closed orbit can be dispensed with. Starting directly from (V.44), we find the invariant

$$C = (Q - p)A + \overline{H^{(1)}} = - \Xi Q^2 \left(\frac{2A}{Q}\right)^{1/2} \cos\psi + (Q - p)A - \frac{n_{xx}}{64}\left(\frac{2A}{Q}\right)^2$$

$$= \frac{R}{2}\left[-2\,\Xi Q^2 \cos\psi + Q(Q - p)R - \frac{n_{xx}}{32} R^3\right]. \qquad (V.46)$$

The fixed points following from this are given by (see section 10)

$$\frac{d\psi}{d\vartheta} = \frac{\partial C}{\partial A} = 0, \quad \frac{dA}{d\vartheta} = -\frac{\partial C}{\partial \psi} = 0,$$

leading to

$$2Q(Q - p) = \pm\, 2\,\Xi Q^2 R^{-1} + \frac{n_{xx}}{8} R^2$$

which is indeed practically identical with (V.42a) (R is positive by definition, whereas $\tilde{c}$ can have either sign). Introduction of the amplitude with respect to one of the fixed points reveals (V.46) identical with (V.45). (V.46) has been used to draw the phase paths in fig. 23 for numerical values of the parameters which might be typical for the CERN PS (the cubic nonlinearity n_{xx} assumed is higher than inherent magnet non-linearities but can can be produced by octupole lenses). The origin of the phase diagram corresponds to the center of the vacuum chamber (or, more generally to the undisturbed equilibrium orbit). The result obtained is independent of which of the closed orbit is used in the calculation. The diagrams correspond to a set of different Q-values around resonance indicated on the non-linear resonance curve in fig. 23.

If there are three fixed points, one is instable, as can be seen from the phase plots. The instable fixed point corresponds to the backward bent branch of the resonance curve in fig. 23 . The amplitude of the corresponding closed orbit is between the amplitudes of the stable closed orbits.

Sufficiently small betatron oscillations have roughly constant amplitude about the stable fixed points (their phase paths are roughly circular about the f.p.). Those getting near the instable fixed point, however, show strong beating and may reach very large amplitudes. It is noteworthy that very large oscillations have constant amplitudes about the undisturbed equilibrium orbit (center of the diagrams) rather than about the distorted closed orbits. From this fact one may conclude that, if all fixed points lie entirely inside the vacuum chamber, the space filled by betatron oscillations does not depend very much on the deformation of the closed orbit (i.e. on the position of the fixed points). In other words : the cubic non-linearity being sufficiently strong, the closed orbit could be allowed to approach the chamber wall to some extent without the particles oscillating about it being peeled off.

The question whether appropriate non-linearities can really be used to reduce the effects of closed orbit distortions is difficult to answer, because the problem is more complicated than in the example considered here (i) by the presence of other perturbation harmonics, and (ii) by dynamic variations of the parameters, in particular of the Q-values. It is, however, likely that non-linearities may help to make the effects of Q-value movements shown in figs.17-22 less alarming.

Appendix VI. Hamiltonian coefficients in terms of machine parameters.

The first approximation Hamiltonian coefficients $v_{\ell_1 m_1 \ell_2 m_2 p}$ needed for the evaluation of the theory are defined by (12.8) :

$$\binom{\ell_1 + m_1}{m_1}\binom{\ell_2 + m_2}{m_2} w^{\ell_1}_{(\vartheta)} \overset{*}{w}{}^{m_1}_{(\vartheta)} u^{\ell_2}_{(\vartheta)} \overset{*}{u}{}^{m_2}_{(\vartheta)} V_{k_1 k_2}(\vartheta) = \sum_{q=-\infty}^{+\infty} v_{\ell_1 m_1 \ell_2 m_2 q}\, e^{-iq\vartheta} , \qquad (12.8 \equiv VI.1)$$

where $V_{k_1 k_2}(\vartheta)$ are the coefficients of the polynomial expansion of the perturbation Hamiltonian in the coordinates x, x', z, z' as given by Appendix (I.24) :

$$H^{(1)} = V_{10}x + V_{01}z + V_{20}x^2 + V_{02}z^2 + V_{11}xz + V_3(x^3-3xz^2) + \ldots \qquad (VI.2)$$

Making use of the Fourier expansions of the Floquet factors w and u in Appendix IV, we obtain Fourier expansions for $w^{\ell_1} \overset{*}{w}{}^{m_1}$, $u^{\ell_2} \overset{*}{u}{}^{m_2}$: :

$$w^{\ell_1} \overset{*}{w}{}^{m_1} = w_0{}^{\ell_1+m_1}\Big\{1 + \frac{\ell_1 w_M + m_1 w_{-M}}{w_0} e^{-iM\vartheta} + \frac{m_1 w_M + \ell_1 w_{-M}}{w_0} e^{iM\vartheta}$$
$$+ \frac{\ell_1 w_{2M} + m_1 w_{-2M}}{w_0} e^{-i2M\vartheta} + \frac{m_1 w_{2M} + \ell_1 w_{-2M}}{w_0} e^{i2M\vartheta}$$
$$+ \frac{\ell_1 w_{3M} + m_1 w_{-3M}}{w_0} e^{-i3M\vartheta} + \frac{m_1 w_{3M} + \ell_1 w_{-3M}}{w_0} e^{i3M\vartheta} + \ldots\Big\}$$

$$u^{\ell_2} \overset{*}{u}{}^{m_2} = w_0{}^{\ell_2+m_2}\Big\{1 - \frac{\ell_2 w_M + m_2 w_{-M}}{w_0} e^{-iM\vartheta} - \frac{m_2 w_m + \ell_2 w_{-M}}{w_0} e^{iM\vartheta}$$
$$+ \frac{\ell_2 w_{2M} + m_2 w_{-2M}}{w_0} e^{-i2M\vartheta} + \frac{m_2 w_{2M} + \ell_2 w_{-2M}}{w_0} e^{i2M\vartheta}$$
$$- \frac{\ell_2 w_{3M} + m_2 w_{-3M}}{w_0} e^{-i3M\vartheta} - \frac{m_2 w_{3M} + \ell_2 w_{-3M}}{w_0} e^{i3M\vartheta} + \ldots\Big\} \qquad (VI.3)$$

Only terms of first order in $\frac{w_{\nu M}}{w_0}$ have been kept here, regarding the fact that $\frac{w_{\nu M}}{w_0} \ll 1$ in practice (see the table in Appendix IV). In many applications, sufficient accuracy is obtained by keeping only the $e^{\pm iM\vartheta}$ terms; as $\frac{w_{\pm 2M}}{w_0}$, ... are small even against $\frac{w_{\pm M}}{w_0}$ in particular examples like the CERN PS.

Applying (VI.3) to (VI.1), the Hamiltonian coefficients can be written :

$$V_{\ell_1 m_1 \ell_2 m_2 q} = \binom{\ell_1+m_1}{m_1}\binom{\ell_2+m_2}{m_2} w_0^{k_1+k_2}\Big[V_{k_1 k_2 q} + \frac{(\ell_1-\ell_2)w_M+(m_1-m_2)w_{-M}}{w_0} V_{k_1 k_2 q-M} + \frac{(m_1-m_2)w_M+(\ell_1-\ell_2)w_{-M}}{w_0} V_{k_1 k_2 q+M}$$

$$+ \frac{(\ell_1-\ell_2)w_{2M}+(m_1-m_2)w_{-2M}}{w_0} V_{k_1,k_2,q-2M} + \frac{(m_1-m_2)w_{2M}+(\ell_1-\ell_2)w_{-2M}}{w_0} V_{k_1,k_2,q+2M} + \dots\Big\} \qquad \text{(VI.4)}$$

where $k_1 = \ell_1+m_1$, $k_2 = \ell_2+m_2$, and $V_{k_1,k_2,q}$ the Fourier coefficients of $V_{k_1 k_2}(\vartheta)$ defined by :

$$V_{k_1 k_2}(\vartheta) = \sum_{q=-\infty}^{+\infty} V_{k_1,k_2,q}\; e^{-iq\vartheta} \qquad \text{(VI.5)}$$

It was pointed out in sections 8 and 12 that in the case of a closed orbit distorted by first order perturbation terms (coefficients V_{10}, V_{01}) new particle coordinates must be introduced by

$$x = c_1 + y_1$$

$$z = c_2 + y_2 \quad . \qquad \text{(VI.6)}$$

y_1, y_2 are the displacements with respect to the perturbed closed orbit $x = c_1(\vartheta)$, $z = c_2(\vartheta)$, which is supposed to be known (e.g. by the method of Appendix V). The Hamiltonian in y_1, y_2, which then contains no terms of order lower than 2, is according to (8.4)

$$\mathcal{H}^{(1)} = H^{(1)}(c + y, c' + y') - \frac{\partial H(c, c')}{\partial c}\; y - \frac{\partial H(c, c')}{\partial c'}\; y'$$

$$= [V_{20}+3c_1 V_{30}+6(c_1^2-c_2^2)V_{40}+ \dots]\, y_1^2 + [V_{02}-3c_1 V_{30}-6(c_1^2-c_2^2)V_{40}+\dots]\, y_2^2$$

$$+ [V_{11}-6c_2 V_{30}-24c_1 c_2 V_{40}\dots]\, y_1 y_2$$

$$+ [V_{30}+4c_1 V_{40}+\dots](y_1^3-3y_1 y_2^2)-[4c_2 V_{40}+\dots](3y_1^2 y_2-y_2^3)$$

$$+ [V_{40}+\dots](y_1^4-6y_1^2 y_2^2 + y_2^4) + \dots$$

$$+ \frac{c_1}{2r_1}\,[(y_1')^2+(y_2')^2] + \frac{y_1}{r_1}\,(c_1' y_1' + c_2' y_2') + \frac{y_1}{2r_1}\,[(y_1')^2+ (y_2')^2] + \dots \qquad \text{(VI.7)}$$

Note that a vertical distortion of the closed orbit ($c_2 \neq 0$) destroys the plane of symmetry of the system and introduces coupling terms, even if $V_{11} = 0$.

The components of the closed orbit may be expressed in terms of the "smooth" closed orbit by (V.2)

$$c_1 = (1 + \rho_1 \cos M\vartheta + ..)\ \bar{c}_1 - (\frac{\sigma_1}{Q} \sin M\vartheta)\ \bar{c}_1' \qquad \text{(VI.8)}$$
$$c_2 = (1 - \rho_1 \cos M\vartheta + ..)\ \bar{c}_2 + (\frac{\sigma_1}{Q} \sin M\vartheta)\ \bar{c}_2'$$

where $\bar{c}_1$, $\bar{c}_2$ can be assumed to contain low order harmonics (<< M) only in appreciable strength. Harmonics will - as before - be designated by adding the appropriate suffix, e.g. c_{1q} and $(c_1^2)_q$ are coefficients of $e^{-iq\vartheta}$ in the Fourier expansions of c_1 and (c_1^2).

The most important of the Hamiltonian coefficients shall now be given in more explicit form. These are

$\underline{v_{11000}, v_{00110},}$ determining the small-oscillation-frequency shifts. They will first be given for the case of an undistorted closed orbit. From (VI.4)

$$\delta Q_1 = v_{11000} = 2\ w_0^2[V_{20,0} + \rho_1(V_{20,-M} + V_{20,M}) + \rho_2(V_{20,-2M} + V_{20,2M}) + ...]$$

$$\delta Q_2 = v_{00110} = 2\ w_0^2[V_{02,0} - \rho_1(V_{02,-M} + V_{02,M}) + \rho_2(V_{02,-2M} + V_{02,2M}) + ...] \qquad \text{(VI.9)}$$

or, by (I.24),

$$v_{11000} = -\frac{2w_0^2}{2!}\ \frac{p_0}{p}\ (\frac{r_m}{r_0})^2 \Big[\ \delta n_0 + \rho_1(\delta n_{-M} + \delta n_M) + \rho_2\ (\delta n_{-2M} + \delta n_{2M}) + ...$$
$$- \frac{\delta p}{p_0}\ (2\rho_1 n_M + 2\rho_3 n_{3M} + ...\ )$$
$$- (\frac{r_0}{r_1})_0 - \rho_2(\frac{r_0}{r_1})_{2M} - ....$$

$$v_{00110} = \frac{2w_0^2}{2!}\ \frac{p_0}{p}(\frac{r_m}{r_0})^2\Big[\delta n_0 - \rho_1(\delta n_{-M} + \delta n_M) + \rho_2(\delta n_{-2M} + \delta n_{2M}) - ...$$
$$+ \frac{\delta p}{p_0}\ (2\rho_1\ n_M + 2\rho_3 n_{3M} + ...)\Big]$$

Here it has been used that $n_{-\nu M} = n_{\nu M} \neq 0$ for odd ν only, and $(\frac{r_0}{r_1})_{-\nu M} = (\frac{r_0}{r_1})_{\nu M} \neq 0$ for even ν only (see Appendix III). Using furthermore (IV.5) or (V.12) this can be written

$$v_{1,000} = - w_0^2 \frac{p_0}{p} \left\{ \left(\frac{r_m}{r_0}\right)^2 \left[\delta n_0 + \rho_1 (\delta n_{-M} + \delta n_M) + \rho_2 (\delta n_{-2M} + \delta n_{2M}) + \dots \right] + 2 Q^2 \frac{\delta p}{p_0} - \frac{r_m}{r_0} \right\}.$$

$$v_{00110} = w_0^2 \frac{p_0}{p} \left\{ \left(\frac{r_m}{r_0}\right)^2 \left[\delta n_0 - \rho_1 (\delta n_{-M} + \delta n_M) + \rho_2 (\delta n_{-2M} + \delta n_{2M}) + \dots \right] - 2 Q^2 \frac{\delta p}{p_0} \right\}.$$

Comments : It had been shown in (IV.8) that very nearly

$$w_0^2 \approx \frac{1}{2Q},$$

so the shift of both Q_1 and Q_2 due to momentum deviations is very nearly

$$\delta Q_1 = \delta Q_2 = - Q \frac{\delta p}{p} .$$

The shift of radial frequency due to non-compensated centrifugal force is

$$\delta Q_1 = w_0^2 \frac{p_0}{p} \frac{r_m}{r_0} \approx \frac{1}{2Q} \frac{r_m}{r_0} .$$

($\left(\frac{r_\rho}{r_1}\right)_0$ had been replaced by its value $\frac{r_\rho}{r_m}$ (III.5); higher harmonics have been omitted in the final formula as the even coefficients ρ_2, ... are very small for the CERN PS).

$\delta n(\vartheta)$ may be due to various causes :

(i) deviation from ideal magnet n-value ($\hat{n} + \delta\hat{n}$ instead of $\hat{n}$, where $\hat{n}$ is the n-value inside the sector centered at $\vartheta = 0$, see Appendix III). In this case we can write

$$\delta n_{\nu M} = \frac{\delta \hat{n}}{\hat{n}} \; n_{\nu M}$$

and

$$\left(\frac{r_m}{r_0}\right)^2 \left[\pm \rho_1 (\delta n_{-M} + \delta n_M) \pm \rho_3 (\delta n_{-3M} + \delta n_{3M}) \pm \dots \right] = \mp 2Q^2 \frac{\delta \hat{n}}{\hat{n}}$$

(ii) Length corrections of magnet sectors. The harmonics produced by these have been calculated in (III.6). Introducing them results in

$$\left[\delta n_0 \pm \rho_1 (\delta n_{-M} + \delta n_M) + \rho_2 (\delta n_{-2M} + \delta n_{2M}) \pm \ldots\right]$$

$$= \frac{\hat{n}M}{\pi}\left\{(\Delta\Theta^F - \Delta\Theta^D)\left[1 + 2\rho_2 \cos 2\frac{\pi}{2}(1 - \frac{r_o}{r_m}) + \ldots\right]\right.$$

$$\left.\pm (\Delta\Theta^F + \Delta\Theta^D)\, 2\left[\rho_1 \cos\frac{\pi}{2}(1 - \frac{r_o}{r_m}) + \rho_3 \cos 3\frac{\pi}{2}(1 - \frac{r_o}{r_m}) + \ldots\right]\right\} .$$

$\Delta\Theta^F$, $\Delta\Theta^D$ are the lengths added to F- and D- magnet ends facing field free sections. If they are length corrections equivalent to fringing fields, we have from Appendix I

$$r_m\, \Delta\Theta^F = r_m\, \Delta\Theta^D = \Delta\ell - \frac{r_o}{n}\,\frac{d\Delta\ell}{dx} \quad .$$

The equivalent length corrections accounting for the field transition across D-F junctions can be disregarded here. Their contribution to δn cancels practically as it is due to opposite influences acting close to the junction.

(iii) Quadrupole lenses : Using the harmonic analysis of δn for this case, made in Appendix III, the frequency shifts due to a set of S quadrupole lenses become

$$\begin{matrix}\delta Q_1 \\ \delta Q_2\end{matrix} = \begin{matrix}v_{11000} \\ v_{00110}\end{matrix} = w_o^2\,\frac{p_o}{p}\left(\frac{r_m}{r_o}\right)^2\left[\mp \delta n_o - \rho_1 (\delta n_{-M} + \delta n_M) \mp \rho_2 (\delta n_{-2M} + \delta n_{2M}) - \ldots\right]$$

$$= w_o^2\,\frac{p_o}{p}\left(\frac{r_m}{r_o}\right)^2 \delta n_o\left[\mp\left(1 + 2\rho_2\,\frac{\sin\frac{2M\eta}{2}}{\frac{2M\eta}{2}} + \ldots\right)\right.$$

$$\left.- (-1)^m\, 2\left(\rho_1\,\frac{\sin\frac{M\eta}{2}}{\frac{M\eta}{2}} + \rho_3\,\frac{\sin\frac{3M\eta}{2}}{\frac{3M\eta}{2}} + \ldots\right)\right]$$

where δn_o is the average produced by the quadrupoles :

$$\delta n_o = \frac{S\eta}{2\pi}\, n_{Quad} = \frac{SL_{quad}}{2\pi r_m}\, n_{Quad} ,$$

and the first lens is located in straight section number m.

Independent variation of Q_1 and Q_2 can be obtained by arranging to sets I and II of quadrupole lenses, located on odd number $(m_2 - m_1)$ of half periods apart. Jointly they produce

$$\begin{matrix}\delta Q_1 \\ \delta Q_2\end{matrix} = w_0^2 \frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \frac{S\eta}{2\pi} \left[\mp (n^I + n^{II}) \left(1 + 2\rho_2 \frac{\sin\frac{2M\eta}{2}}{\frac{2M\eta}{2}} + ..\right) \right.$$

$$\left. - (-1)^{m_1} \quad (n^I - n^{II}) \; 2 \left(\rho_1 \frac{\sin\frac{M\eta}{2}}{\frac{M\eta}{2}} + \rho_3 \frac{\sin\frac{3M\eta}{2}}{\frac{3M\eta}{2}} + ..\right) \right] .$$

In the (Q_1, Q_2) diagram, a variation of $(n^I - n^{II})$ moves Q_1, Q_2 along the main diagonal, and a variation of $(n^I + n^{II})$ perpendicular to it.

The maximum quadrupole strength required to shift Q by δQ in the CERN PS is roughly given by

$$\frac{SL}{2\pi r_0} n_{Quad} \approx 16 \; \delta Q \quad (L = \text{length of quadrupole lens})$$

or

$$\frac{1}{B_0} \frac{\partial B}{\partial r} \approx \frac{2\pi . 16 \; \delta Q}{SL} .$$

It must be emphasized that the foregoing first approximation formulae for the effect of quadrupole lenses are very noticeably modified in second approximation, if S is a small fraction of M (see further below).

In the case of a distorted closed orbit, (VI.9) has to be modified according to (VI.7). In working out the harmonics of the coefficients, we remember (VI.8) and find

$$c_{10} = \bar{c}_{10} \;, \qquad c_{1,\pm M} = \frac{\rho_1}{2} \bar{c}_{10} \;; \qquad c_{1\pm 2M} = \frac{\rho_2}{2} \bar{c}_{10}$$

$$(c_1^2 - c_2^2)_0 = (\bar{c}_1^2 - \bar{c}_2^2)_0 \;; \; (c_1^2 - c_2^2)_{\pm M} = \rho_1 \; (\bar{c}_1^2 + \bar{c}_2^2)_0 \; .$$

In the second line, squares of $\rho_1, \rho_2 ..$ have been neglected; also terms involving $\bar{c}_1'$ and $\bar{c}_2'$. These approximations restrict the following formulae to closed orbits which are not too wavy.

Thus we obtain instead of (VI.9), neglecting ρ's in powers higher than 1, and making use of $V_{k_1 k_2(-M)} = V_{k_1 k_2(M)}$ (always holding in our case)

$$\delta Q_1 = v_{11000} = 2w_0^2 \Big\{ [V_{20} + 3c_1 V_{30} + 6(c_1^2 - c_2^2) V_{40} + ..]_0$$

$$+ 2\rho_1 [V_{20} + 3c_1 V_{30} + 6(c_1^2 - c_2^2) V_{40} + ..]_M$$

$$+ 2\rho_2 [V_{20} + 3c_1 V_{30} + 6(c_1^2 - c_2^2) V_{40} + ..]_{2M} + .. \Big\}$$

$$= 2w_0^2 \Big\{ \Big[V_{20,0} + 3(c_{10} V_{300} + 2c_{1M} V_{30M} + ..) + 6\Big((c_1^2 - c_2^2)_0 V_{400} + 2(c_1^2 - c_2^2)_M V_{40M} + ..\Big)\Big]$$

$$+ 2\rho_1 [V_{20M} + 3c_{10} V_{30M} + 6(c_1^2 - c_2^2)_0 V_{40M} + ..]$$

$$+ 2\rho_2 [V_{20,2M} + 3c_{10} V_{30,2M} + 6(c_1^2 - c_2^2)_0 V_{40,2M} + ..] + .. \Big\}$$

$$= 2w_0^2 \Big\{ [V_{20,0} + 2\rho_1 V_{20,M} + 2\rho_2 V_{20,2M} + ..]$$

$$+ 3\overline{c_{10}} [V_{30,0} + 3\rho_1 V_{30M} + 3\rho_2 V_{30,2M} + ..]$$

$$+ 6(\bar{c}_{10}^2 - \bar{c}_{20}^2) V_{40,0} + 6(\bar{c}_{10}^2) 4[\rho_1 V_{40,M} + \rho_2 V_{40,2M} + ..] + \ldots \Big\}$$

$$= - w_0^2 \frac{p_0}{p} \Big\{ (\frac{r_m}{r_0})^2 [\delta n_0 + 2\rho_1 \delta n_M + 2\rho_2 \delta n_{2M} + ..] + 2Q^2 \frac{\delta p}{p_0} - \frac{r_m}{r_0}$$

$$+ \overline{c_{10}} (\frac{r_m}{r_0})^2 \Big[(\frac{dn}{dx})_0 + 3\rho_1 (\frac{dn}{dx})_M + .. \Big]$$

$$+ \frac{1}{2} (\bar{c}_{10}^2 - \bar{c}_{20}^2)(\frac{r_m}{r_0})^2 (\frac{d^2n}{dx^2})_0 + 2\, \bar{c}_{10}^2 (\frac{r_m}{r_0})^2 \Big[\rho_1 (\frac{d^2n}{dx^2})_M + .. \Big] \Big\}$$

$$\delta Q_2 = v_{00110} = 2w_0^2 \Big\{ [V_{02} - 3\, c_1 V_{30} - 6(c_1^2 - c_2^2) V_{40} + ..]_0$$

$$- 2\rho_1 [V_{02} - 3\, c_1 V_{30} - 6(c_1^2 - c_2^2) V_{40} + ..]_M$$

$$+ 2\rho_2 [\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad]_{2M} \pm .. \Big\}$$

$$= + w_0^2 \frac{p_0}{p} \Big\{ (\frac{r_m}{r_0})^2 [\delta n_0 - 2\rho_1 \delta n_M + 2\rho_2 \delta n_{2M} \mp \ldots] - 2Q^2 \frac{\delta p}{p_0}$$

$$+ \bar{c}_{10} (\frac{r_m}{r_0})^2 \Big[(\frac{dn}{dx})_0 - \rho_1 (\frac{dn}{dx})_M + 3\rho_2 (\frac{dn}{dx})_{2M} \ldots \Big]$$

$$+ \frac{1}{2} (\bar{c}_{10}^2 - \bar{c}_{20}^2)(\frac{r_m}{r_0})^2 (\frac{d^2n}{dx^2})_0 + 2\bar{c}_{10}^2 (\frac{r_m}{r_0})^2 \Big[\rho_2 (\frac{d^2n}{dx^2})_{2M} + .. \Big]$$

$$+ 2\bar{c}_{20}^2 (\frac{r_m}{r_0})^2 \Big[\rho_1 (\frac{d^2n}{dx^2})_M + \rho_3 (\frac{d^2n}{dx^2})_{3M} + .. \Big] \Big\}$$

(It can be verified that these formulae agree with what would be obtained for δQ_1 and δQ_2 from (V22,23), apart from those higher harmonics of $\frac{dn}{dx}$ and $\frac{d^2n}{dx^2}$ which had been neglected in deriving (V22,23)). Those parts of δn and $\frac{d^2n}{dx^2}$ which are due to the alternating magnets (and not due to, e.g., lenses) can be summed up according to

$$(\frac{r_m}{r_o})^2\ [\rho_1\ \delta n_M + \rho_3\ \delta n_{3M} + ..] = -\ Q^2\ \frac{\delta\hat{n}}{\hat{n}}$$

$$(\frac{r_m}{r_o})^2\ [\rho_1\ (\frac{d^2n}{dx^2})_M + \rho_3\ (\frac{d^2n}{dx^2})_{3M} + ..] = -\ \frac{Q^2}{\hat{n}}\ \frac{d^2\hat{n}}{dx^2}\ ,$$

where $\frac{d^2\hat{n}}{dx^2}$ is the derivative at $x = z = 0$ in the sector centered at $\vartheta=0$.

Omitting the effect of lenses and other additional influences, and using the approximations $w_0^2 \approx \frac{1}{2Q}$, $\frac{p_o}{p} \approx 1$ the formulae become

$$\delta Q_1 \approx Q\ (\frac{\delta\hat{n}}{\hat{n}} - \frac{\delta p}{p_o}) + \frac{1}{2Q}\ \frac{r_m}{r_o} - \bar{c}_{10}\ \frac{1}{2Q}\ (\frac{r_m}{r_o})(\frac{d\hat{n}}{dx}) + \bar{c}_{10}^2\ \frac{Q}{\hat{n}}\ (\frac{d^2\hat{n}}{dx^2})$$

$$\delta Q_2 \approx Q\ (\frac{\delta\hat{n}}{\hat{n}} - \frac{\delta p}{p_o}) + \bar{c}_{10}\ \frac{1}{2Q}\ \frac{r_m}{r_o}\ \frac{d\hat{n}}{dx} - \bar{c}_{20}^2\ \frac{Q}{\hat{n}}\ \frac{d^2\hat{n}}{dx^2}\ .$$

<u>v_{22000}, v_{00220}, v_{11110}</u> are the coefficients determining the shift of frequency with amplitude. From (VI.4) and (I.24)

$$v_{22000} = 6\ w_0^4\ [V_{40,0} + 2\rho_1(V_{40(-M)} + V_{40(M)}) + ..]$$

$$= -\ \frac{6w_0^4}{4!}\ \frac{p_o}{p}\ (\frac{r_m}{r_o})^2\ [(\frac{d^2n}{dx^2})_o + 4\rho_1(\frac{d^2n}{dx^2})_M + ..]$$

$$v_{00220} = 6w_0^4\ [V_{40,0} - 2\rho_1(V_{40(-M)} + V_{40(M)}) + ..]$$

$$= -\ \frac{6w_0^4}{4!}\ \frac{p_o}{p}(\frac{r_m}{r_o})^2\ [(\frac{d^2n}{dx^2})_o - 4\rho_1(\frac{d^2n}{dx^2})_M + ..]$$

$$v_{11110} = 4\ w_0^4\ [V_{220} + 2\rho_2(V_{22(-2M)} + V_{22(2M)}) + 2\rho_4\ (...)]$$

$$= -\ 4\ w_0^4\ .\ 6\ [V_{40,0} + ..]$$

$$= w_0^4\ \frac{p_o}{p}(\frac{r_m}{r_o})^2\ [(\frac{d^2n}{dx^2})_o + 4\rho_2\ (\frac{d^2n}{dx^2})_{2M} + ..]$$

If these coefficients are solely due to the alternating magnets they can be written

$$v_{22000} = - v_{00220} = \frac{1}{4\hat{n}} \frac{d^2\hat{n}}{dx^2}$$

$$v_{11110} = 0 \; .$$

Coefficients responsible for the excitation of resonances :

The terms exciting first order resonances $(q \approx Q)$ are discussed in Appendix V.

Second order resonances $(q \approx 2Q)$ are excited by

$$v_{2000q} = w_0^2 \left[V_{20,q} + 2 \frac{w_M}{w_0} V_{20,q-M} + 2 \frac{w_{-M}}{w_0} V_{20,q+M} + \dots \right]$$

or, in the case of a distorted closed orbit

$$v_{2000q} = w_0^2 \left[V_{20q} + 3\left(c_1 V_{30}\right)_q + 6 \left((c_1^2 - c_2^2) V_{40} \right)_q + \dots \right]$$

$$= - \frac{1}{2} w_0^2 \frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \left[\delta n_q + \left(c_1 \frac{dn}{dx} \right)_q + \frac{1}{2} \left((c_1^2 - c_2^2) \frac{d^2 n}{dx^2} \right)_q + \dots \right] .$$

Terms multiplied by $\frac{w_M}{w_0}$ ... etc. have not been written down in the second line as they will in general be smaller than the preceding terms; moreover, only order-of-magnitude guesses are possible anyway in practice.

$$v_{0020q} = w_0^2 \left[V_{02q} - 3(c_1 V_{30})_q - 6 \left((c_1^2 - c_2^2) V_{40} \right)_q + \dots \right] = - v_{2000q}$$

$$v_{1010q} = w_0^2 \left[V_{11q} - 6(c_2 V_{30})_q - 24(c_1 c_2 V_{40})_q \dots \right]$$

$$= - w_0^2 \frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \left[2(\epsilon n)_q - \left(c_2 \frac{dn}{dx}\right)_q - \left(c_1 c_2 \frac{d^2 n}{dx^2}\right)_q \right]$$

Comments : Harmonics, whose order is not a multiple of the number M of magnet periods or S of superperiods are due to :

(i) fluctuations of n (harmonics : δn_q);

(ii) perturbations of the median plane by twisted magnets. The harmonics $(\epsilon n)_q$ can be calculated analogously to $(\xi n)_q$ by (V.24). For random twists ($\sqrt{\langle \Delta\epsilon^2 \rangle}$ r.m.s. angle per magnet unit) and order $q \ll M$, the m.s. value, found by the same procedure as (V.32), is

$$< |(\epsilon n)_q|^2 > = \frac{1}{2M} (\frac{\hat{n}}{4})^2 < \Delta\epsilon^2 >$$

(iii) Combination of closed orbit distortions with non-linearities; e.g. c_q may combine with $(\frac{dn}{dx})_o$, or more generally $c_{q-\nu S}$ with $(\frac{dn}{dx})_{\nu S}$ (in the presence of sextupole lenses arranged in S superperiods) to form q-th harmonic perturbations.

3rd order resonances are excited by

$$v_{3000q} = w_o^3 \; [V_{30} + 4(c_1 V_{40})_q + ..] = - \frac{w_q^3}{3!} \frac{p_o}{p} (\frac{r_m}{r_o})^2 \left[(\frac{dn}{dx})_q + (c_1 \frac{d^2n}{dx^2})_q + .. \right]$$

$$v_{0030q} = \frac{w_o^3}{3!} \frac{p_o}{p} (\frac{r_m}{r_o})^2 \left[3 \; (\epsilon \frac{dn}{dx})_q - (c_2 \frac{d^2n}{dx^2})_q + .. \right]$$

$$v_{2010q} = - 3 \; v_{0030q}$$

$$v_{1020q} = - 3 \; v_{3000q}$$

4th order resonances are excited by

$$v_{4000q} = v_{0040q} = w_o^4 \; [V_{40q} + ..] = - \frac{w_o^4}{4!} \frac{p_o}{p} (\frac{r_m}{r_o})^2 (\frac{d^2n}{dx^2})_q + ..$$

$$v_{2020q} = - 6 \; v_{4000q}$$

$$v_{3010q} = - v_{1030q} = - \frac{w_o^4}{3!} \frac{p_o}{p} (\frac{r_m}{r_o})^2 \; 4(\epsilon \frac{d^2n}{dx^2})_q$$

Comments : Harmonics of order q different from νM or νS are due to

(i) fluctuation of $\frac{dn}{dx}, \frac{d^2n}{dx^2}$, ...

(ii) combination harmonics of a distorted closed orbit and derivatives of n. Such combination harmonics occur of course also with non-linearities of higher orders than those written down, if there are any present.

(iii) Fluctuating twists ϵ : Some of the coefficients appear only on account of such twists. With unperturbed plane of symmetry, the corresponding resonances would not be excited, as

$$Q_1 \pm Q_2 = q$$

$$3Q_2 = q; \qquad 2Q_1 \pm Q_2 = q$$
$$3Q_1 \pm Q_2 = q, \qquad Q_1 \pm 3Q_2 = q$$

for resonance orders 2, 3, 4.

In second approximation, the coefficients $v_{\ell_1 m_1 \ell_2 m_2 q}$ have to be replaced by corrected ones $h_{\ell_1 m_1 \ell_2 m_2 q}$ given by (6.9) for one-dimensional motion. In most applications of the theory in the present report, the second approximation corrections have been neglected. The justification of this (for CERN PS parameters) may be checked for the coefficients determining frequency shifts (see (6.10)). Taking account of systematic perturbations having the magnet period only, we have

$$\delta Q_1 = h_{110} = v_{110} - 4 \sum_{q=0,\pm M,\pm 2M,\dots} \frac{|v_{20q}|^2}{2Q - q}$$

with (using (VI.4) and (I.24))

$$v_{200} = w_0^2 \left[V_{2(0)} + 2\rho_1 V_{2(M)} + ..\right] \approx - \frac{1}{2Q} \frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \frac{1}{2} \left[2\rho_1 \, \delta n_M + ..\right]$$

$$v_{20M} = w_0^2 \left[V_{2(M)} + 2\frac{w_M}{w_0} V_{2(0)} + 2\frac{w_{-M}}{w_0} V_{2(2M)} + ..\right] \approx - \frac{1}{2Q} \frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \frac{1}{2} \left[\delta n_M + ..\right]$$

$$v_{20(2M)} = w_0^2 \left[V_{2(2M)} + 2\frac{w_M}{w_0} V_{2(M)} + 2\frac{w_{-M}}{w_0} V_{2(3M)} + ..\right] \approx - \frac{1}{2Q} \frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \frac{1}{2} \left[2\frac{w_M}{w_0} \delta n_M + ..\right]$$

giving

$$\sum \approx \frac{1}{4} \left(\frac{p_0}{p} \left(\frac{r_m}{r_0}\right)^2 \rho_1 \, \delta n_M\right)^2 \frac{1}{Q^3} \left[\frac{1}{2} + \frac{1}{\rho_1^2} \frac{Q^2}{(2Q)^2 - M^2} + \left(\frac{2}{\rho_1} \frac{w_M}{w_0}\right)^2 \frac{Q^2}{(2Q)^2 - (2M)^2}\right]$$

$$\approx \frac{1}{4} \left(\frac{\delta \hat{n}}{\hat{n}}\right)^2 Q \left[0.5 - 0.59 - 0.005 \, ..\right] = \approx - \frac{0.09}{4} \left(\frac{\delta \hat{n}}{\hat{n}}\right)^2 Q,$$

or

$$\delta Q = Q \frac{\delta \hat{n}}{\hat{n}} \left[1 + 0.09 \frac{\delta \hat{n}}{\hat{n}}\right].$$

Thus, for a change of Q of 10%, the first approximation is wrong by only about 1%.

The second approximation is more noticeable in the calculation of the effect of quadrupole lenses, because lower harmonic numbers q appear in the correction term (i.e. multiples of the number of superperiods).

As to the non-linear frequency shift determined by h_{220} (bottom line of (6.10)), there appears a contribution of the quadratic non-linearity in the second approximation. Supposing first that the quadratic non-linearity is due to $\frac{dn}{dx}$ in the magnetic guide field, only the zero harmonic q = 0 is important, and we get

$$h_{220} = v_{220} - 3\left[3\,\frac{|v_{300}|^2}{3Q} + \frac{|v_{210}|^2}{Q}\right] = v_{220} - \frac{30}{Q}\,|v_{300}|^2$$

$$= -\frac{n_{xx}}{16Q^2} - \frac{5}{48}\,\frac{(n_x)^2}{Q^4}$$

with

$$v_{300} = -\frac{w_0^3}{3!}\,\frac{p_0}{p}\,\left(\frac{r_m}{r_0}\right)^2\,\left(\frac{dn}{dx}\right)_0 \approx -\frac{n_x}{6(2Q)^{3/2}}$$

$$v_{210} = 3v_{300}$$

$$n_x = \left(\frac{r_m}{r_0}\right)^2\left(\frac{dn}{dx}\right)_0 = \frac{r_m}{r_0}\,\frac{d\hat{n}}{dx}; \qquad n_{xx} = \left(\frac{r_m}{r_0}\right)^2\left[\left(\frac{d^2n}{dx^2}\right)_0 + 4\rho_1\left(\frac{d^2n}{dx^2}\right)_M + ..\right] = -\frac{4Q^2}{\hat{n}}\,\frac{d^2\hat{n}}{dx^2}$$

At low and high fields (uncorrected) in the CERN PS the contribution to h_{220} by n_x may equal or predominate that by n_{xx}.

Considering now a quadratic perturbation produced by sextupole lenses, its contribution to h_{220} may be enhanced if small denominators (3Q-q) appear in (6.10). For the CERN PS structure for example, with S = 10 superperiods this would happen for q = 20, unless the coefficient $v_{30(2S)}$ is zero.

The terms in the perturbation Hamiltonian depending on the derivatives (last line in (I.24)) have been neglected in the present calculation of Hamiltonian coefficients. Using numerical parameters which are typical for the CERN PS, those terms are formed to be small in comparison with the preceding terms (which depend on the coordinates only). We check the influence of the third degree term $\frac{x}{r_1}[(x')^2 + (z')^2]$, contributing to the quadratic non-linearity in the equations of motion : the first noticeable effect would be on the shift of Q with displacement of the closed orbit. Therefore, we calculate the contribution to h_{11000} of the corresponding term in the Hamiltonian (VI.7) for a displaced closed orbit, i.e. of

$$\frac{1}{2r_1} [c_1(y_1')^2 + c_1(y_2')^2 + 2c_1' y_1 y_1' + 2c_2' y_1 y_2'] .$$

After transformation to a_1, φ_1, a_2, φ_2 according to (12.5) (12.6), we find

$$h_{11000} = \frac{1}{2r_m}[2c_1 w_2 w_2^* + c_1'(w_1 w_2^* + w_1^* w_2)]_0 \approx \bar{c}_1 \; w_0^2 \; \frac{3}{2} \frac{(M\rho_1)^2}{r_m} ,$$

having used (II.15), (V.1), (V.2) and (VI.8). Comparing this with v_{11000} as given earlier in this appendix, the effect is seen to be equivalent to a $\frac{dn}{dx}$ following from :

$$\frac{p_s}{p} \left(\frac{r_m}{r_0}\right)^2 \left(\frac{dn}{dx}\right)_0 = - \frac{3}{2} \frac{(M\rho_1)^2}{r_m} = - 0.011 \; [\mathrm{cm}^{-1}] .$$

This is indeed an extremely small non-linearity.

The same result was also found in appendix V.

<u>The polynomial expansion of n(x)</u>, on which the perturbation analysis in the present paper is based, requires some comment : In practice, the expansion must be limited to a small number of terms. The formulae in this appendix do not contain derivatives higher than $\frac{d^2n}{dx^2}$, thus corresponding to

$$n = n(0) + \left(\frac{dn}{dx}\right)_{x=0} x + \frac{1}{2} \left(\frac{d^2n}{dx^2}\right)_{x=0} x^2 .$$

When approximating a given function $n(x)$ by a finite polynomial, the relevant first terms of the Taylor expansion of $n(x)$ do in general not provide the optimum approximation. This is rather obtained by a "least square" fit of the polynomial to the given function within the interval x of interest (i.e. the aperture of the vacuum chamber). For illustration, 4th degree and 2nd degree least square fits of $n(x)$-curves of the CERN PS are shown in fig. 24 . Furthermore, a 2nd degree polynomial, comprising only the first 3 terms of the fourth degree fit, has been plotted. Although giving a better approximation for small x, the latter 2nd degree polynomial is far off for larger x, whereas the least square 2nd degree polynomial keeps closer to the given $n(x)$.

There is a difficulty which needs some clarification : If a finite least square polynomial expansion is used to represent also the part of $n(x)$

fluctuating with ϑ, this expansion may contain powers which do not appear in the rigorous Taylor expansion, for instance an x term which may be missing in the rigorous expansion. Correspondingly a 3rd order subresonance could be excited by the approximation term which at first sight would not be excited by the rigorous expansion. Looking, however, at (9.2) we find the general excitation term to be (specialized to the third order subresonance)

$$\left(h_{30p}\, A^{3/2} + h_{41p}\, A^{5/2} + h_{52p}\, A^{7} + \dots\right) e^{i3[(Q-\frac{p}{3})\vartheta + \phi]} + \text{conj. complex}$$

$$\approx - \left[\frac{1}{3!}\left(\frac{dn}{dx}\right)_p (w_0 A^{1/2})^3 + \frac{5}{5!}\left(\frac{d^3n}{dx^3}\right)_p (w_0 A^{1/2})^5 + \dots\right] e^{i3[(Q - \frac{p}{3})\,\vartheta + \phi]} + \text{conj. complex}$$

So actually all odd powers (>3 in the Hamiltonian or > 1 in $n(x)$) are contributing to the excitation (for an even order subresonance it would be the even powers). Using a least square fit limited to the derivative $\frac{d^2n}{dx^2}$ at the maximum means replacing the excitation term by the simpler one $h_{30p}\, A^{3/2} \approx - \frac{1}{3!}(\frac{dn}{dx})_p\, (w_0\, A^{1/2})^3$, which approximates the more complicated dependence on amplitude in the average. The results obtained from this approximation can be expected to give fairly correct overall beating ratios and beating periods, whereas the approximation be less good for the details of the motion.

There is some <u>contribution to non-linearities by fringing fields</u> at magnet edges and junctions. Expressing the perturbation of B by fringing fields in terms of equivalent length $\Delta\ell$ of the ideal field as explained in Appendix I, the integrated contribution of one boundary to δn is according to (I.20)

$$\int \delta n\; d\vartheta = - \frac{r_0}{B_0}\frac{d}{dx}\left[(1 - \frac{nx}{r_0})\; B_0\; \frac{\Delta\ell(x)}{r_m}\right] = \frac{n}{r_m}\left[\Delta\ell - \frac{r_0}{n}\frac{d\Delta\ell}{dx} + x\frac{d\Delta\ell}{dx}\right] = \frac{n\Delta\ell_G}{r_m}$$

$\Delta\ell_G$ is for $x = 0$ the equivalent length for gradient introduced earlier. From this the integrated contribution to $\frac{dn}{dx}$ follows as

$$\int \frac{dn}{dx}\, d\vartheta = \frac{n}{r_m}\,\frac{d\Delta\ell_G}{dx} = \frac{2n}{r_m}\,\frac{d\Delta\ell}{dx} - \frac{r_0}{r_m}\,\frac{d^2\Delta\ell}{dx^2}$$

The contribution to the quadratic non-linearity is the most important one. Its zeroth harmonic $(\frac{dn}{dx})_0$ becomes

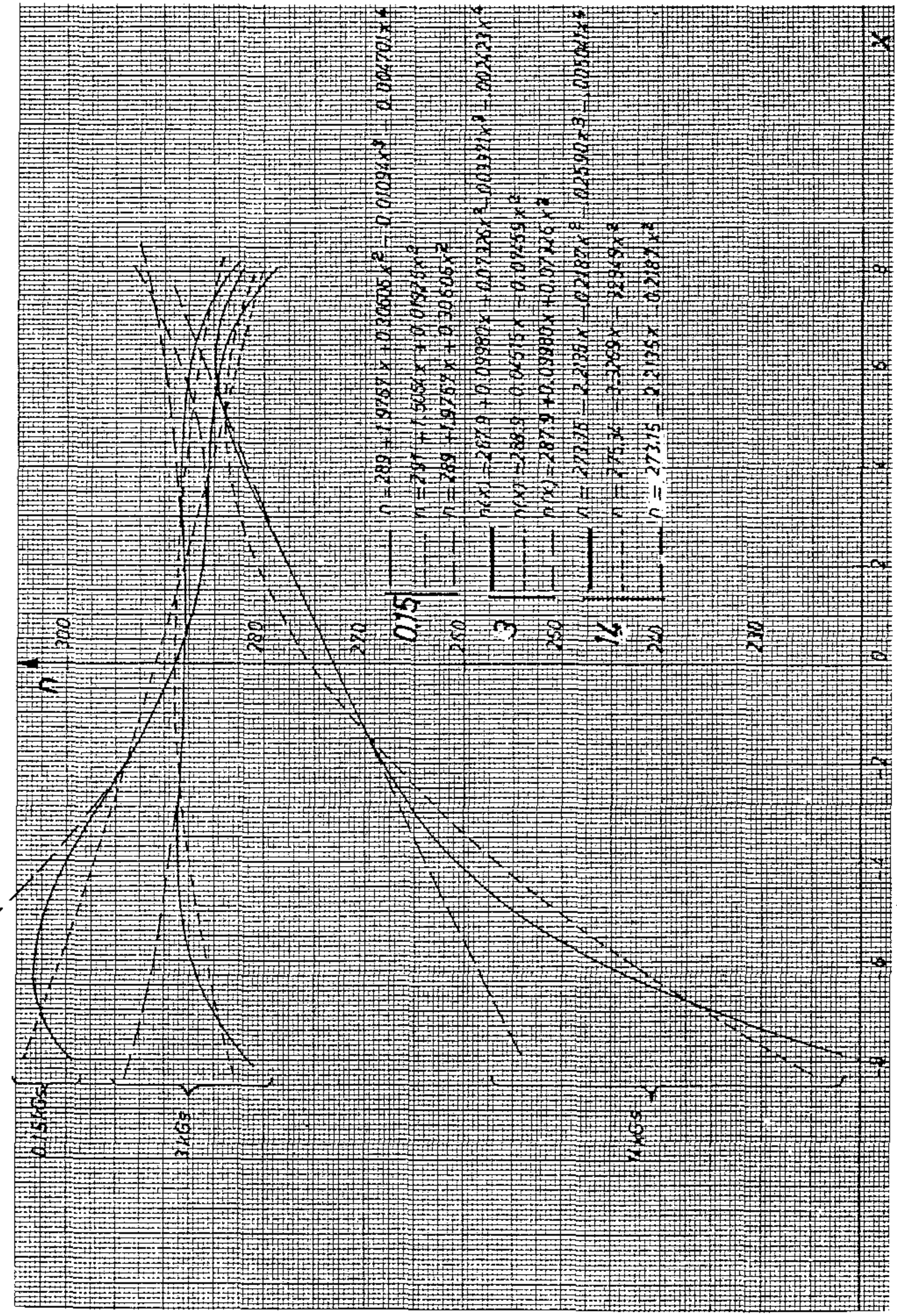

Fig. 24 Approximation of empirical n(x) curves by 4th order and second order polynomials. The 4th order approximation (full line) represents measured values fairly accurately.

$$\left(\frac{dn}{dx}\right)_o^{fringing\ fields} = \frac{4M}{2\pi}\frac{n}{r_m}\left[\left(\frac{d\Delta\ell_G}{dx}\right)_{ends} + \left(\frac{d\Delta\ell_G}{dx}\right)_{junction\ boundaries}\right].$$

In the CERN PS roughly the following contributions to $(\frac{dn}{dx})_o$ are found

	B_o =	150	3000	14000	Gauss
magnets	$\left(\frac{dn}{dx}\right)_o$ =	- 1.4	- 0.07	1.4	cm^{-1}
end faces	$\left(\frac{dn}{dx}\right)_o$ =	0.27	0.27	0.36	cm^{-1}
junctions	$\left(\frac{dn}{dx}\right)_o$ =	- 0.31	- 0.31	0	cm^{-1}
total	$\left(\frac{dn}{dx}\right)_o$ =	- 1.4	- 0.11	1.8	cm^{-1}

Except for high fields, end and junction effects practically cancel.

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List of references

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Adams, J.B. and Hine, M.G.N. [1953a]. Notes on the results obtained from the computing program investigating the effects of non-linear magnetic fields on the build-up of betatron oscillations by azimuthal inhomogenities.

CERN PS/JBA-MG.NH 1*, ... 2, 47, 62, 122

Adams, J.B. and Hine, M.G.N. [1953b]. Further notes on the results of the computing program investigating the effects of non-linear magnetic fields on the betatron oscillations in a strong focussing proton synchrotron.

CERN-PS/JBA-MGNH 2*, ... 2, 47

Adams, J.B. and Hine, M.G.N. [1953c]. Theory of the non-linear system used in the stage 1 computation.

CERN-PS/JBA-MGNH 3*, ... 2, 47

Adams, J.B. and Hine, M.G.N. [1953d]. The choice of magnet sector arrangement in the strong focussing synchrotron.

CERN-PS/JBA-MGNH 17*, ... 2, 47, 62, 122

Adams, J.B. and Hine, M.G.N. [1954]. The variations of the parameters of an A.G. synchrotron with μ and n.

CERN PS/JBA-MGNH 22* 62

Adams, J.B. and Hine, M.G.N. [1956]. Variation in the Q value of a synchrotron with radial displacement of the orbit.

CERN-PS/JBA-MGNH 25* 61

Bell, J.S. [1953]. Basic algebra of the strong focussing system. (AERE T/R 1114).

Atomic Energy Research Establishment, Harwell........................ 2, 75

Bell, M. [1955]. Non-linear equations of motion in the synchrotron. (AERE-T/M 125).

Atomic Energy Research Establishment, Harwell 75

Beth, H.J.E. [1910]. The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations.

Koninklijke Akademie van Wetenschappen, Amsterdam. B. Verhandelingen. v.12, p. 619-34, 735-50, ... 3, 23

* unpublished, but available for consultation at main accelerator centres.

CERN-PS. Magnet measurement group [1957].
CERN PS/MM-Int* 9 60

Cole, F.T. [1954]. Perturbation theory of A.G. motion with non-linear restoring forces. (MURA-FTC-1).
Midwestern Universities Research Association, Madison 4

Courant, E.D. [1949]. A resonance effect in the synchrotron.
Journal of applied physics, 20, p. 611-6 2

Courant, E.D. [1956]. Non-linearities in the A.G. synchrotron.
In CERN Symposium 1956. Proceedings, v. 1, p. 254-61 4

Courant, E.D. Livingston, H.S. and Snyder, H.S. [1952]. The strong-focusing synchrotron - a new high energy accelerator.
Physical Review, 88, p. 1190-6, ... 1

Courant, E.D. Livingston, M.S., Snyder, H.S. and Blewett, J.P.[1953]. Origin of the "strong-focusing" principle.
Physical Review, 91, p. 202-3, ... 1

Dennison, D.M. and Berlin, T.H. [1946]. The stability of orbits in the racetrack.
Physical Review, 69, p. 542-3, ... 2

Hagedorn, R. [1955]. Stability of two-dimensional non-linear oscillations with periodical Hamiltonian; applied to the non-linear betatron oscillations of an alternating-gradient synchrotron.
CERN-PS/RH-9* 3, 45

Hagedorn, R. [1956]. Stability of two-dimensional non-linear oscillations with periodical Hamiltonian, applied to the non-linear betatron oscillations of circular particle accelerators.
CERN-PS/RH 10* 3

Hagedorn, R. [1957]. Stability and amplitude ranges of two dimensional non-linear oscillations with periodical Hamiltonian applied to betatron oscillations in circular particle accelerators. Parts I and II.
CERN 57-1 (CERN Reports. Geneva, 1957) 45

Hagedorn, R. Hine, M.G.N. and Schoch, A. [1956]. Non-linear orbit problems in synchrotrons.
In CERN Symposium 1956. Proceedings. v. 1, p. 237-253 1

Hine, M.G.N. [1953]. Topics on non-linear orbit theory.
In lectures on the theory and design of an alternating-gradient proton synchrotron. Geneva, CERN, p. 69-81 4

* See erratum

Johnsen, K. [1956]. Effects of non-linearities on the phase transition.
In CERN Symposium 1956. Proceedings. v.1, p. 106-111 66

Judd, D.L. [1950]. A study of the injection process in synchrotrons and betatrons.
Thesis : California Institute of Technology, ... 3, 23

Kerst, D.W. and Serber, R.[1941]. Electronic orbits in the induction accelerator.
Physical Review, 60, p. 53-8, ... 1

Kolomenski, A.A. [1956]. On the non-linear theory of betatron oscillations.
In CERN Symposium 1956. Proceedings, v. 1, p. 265-78 4

Krylov, N.M. and Bogoliubov, N.N. [1937]. Introduction to non-linear mechanics).
Kiev, 1937 .... 3, 23

Lüders, G. [1953a]. Line theory of betatron oscillations.
In Lectures on the theory and design of an alternating-gradient proton-synchrotron. Geneva, CERN, 1953. p. 45-67 2

Lüder, G. [1953b]. Statistical analysis of closed orbit and stop bands.
CERN-PS/GL 8* 62, 122

Lüders, G. [1955]. Über den Einfluss von Fehlern des magnetischen Feldes auf die Betatronschwingungen in Synchrotron mit starker Stabilisierung.
Nuov. Cim. Suppl., 2, p. 1075-146, ... 2, 62

Lüders, G. [1956]. On the influence of irregularities of magnetic field on betatron oscillations in an alternating gradient synchrotron.
CERN 56-8 (CERN Reports, Geneva, 1956) 2, 62

Moser, J. [1955]. Stabilitätsverhalten kanonischer Differentialgleichungssysteme.
Nachrichten der Akademie der Wissenschaften in Göttingen. IIa. No. 6, p. 87-120, ... 2, 11, 23, 26, 32

Parzen, G. [1956]. Non-linear resonances in alternating gradient accelerators. (MURA-200).
Midwestern Universities Research Association, Madison 4

Parzen, G. [1957]. Coupled non-linear resonances in alternating gradient accelerators (MURA-217).
Midwestern Universities Research Association, Madison .... 4.

Powell, J.L. and Wright, R.S. [1955]. Non-linearities in A.G. synchrotrons. (MURA - RW/JLP 4).

Midwestern Universities Research Association, Madison, ... 2

Schoch, A. [1955]. Orbit stability in a synchrotron with non-linear restoring forces.

CERN-PS/A.Sch. 2* 3

Sturrock, P.A. [1955]. Non-linear effects in alternating gradient synchrotrons (AERE-X/R1771).

Atomic Energy Research Establishment, Harwell, ... 2, 23

Symon, K.R. [1954]. A smooth approximation to the alternating gradient orbit equations. (MURA-KRS-1).

Midwestern Universities Research Association, Madison, ... 4

Symon, K.R. [1954b]. Smooth solution to one-dimensional AG orbits with cubic forces (MURA-KRS-2).

Midwestern Universities Research Association, Madison 4

Symon, K.R. and Sessler, A.M. [1956]. Methods of radio frequency acceleration in fixed field accelerators.

<u>In</u> CERN Symposium 1956. Proceedings. v.1, p. 44-58 54

Whittaker, E.T. [1937]. A treatise on the analytical dynamics of particles and rigid bodies. 4th ed.

Cambridge, University press.. 75

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Proton Synchrotron Division

1st February 1958

ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE

CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

Theory of linear and non-linear perturbations of betatron oscillations in alternating gradient synchrotrons

by

A. Schoch

GENEVE

Erratum

The following report

> Schoch, A. Theory of linear and non-linear perturbations of betatron oscillations in alternating gradient synchrotrons

was given in error the number CERN 57-23. Would you please change this number to

CERN 57-21

on your copy.

* * *

The following items should also be added to this report :

p.68 a footnote reading :

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* It should be noted that "smooth motion" values have been used for the closed orbit harmonics. Rigorously the wriggle modulation of the closed orbit by the magnet periodicity can combine with the magnet periodicity to produce exciting harmonics without lenses, which would be, however, about an order of magnitude smaller than those given above.

p.151 a reference as follows :

Hagedorn R. and Schoch A. (1957). Stability and amplitude ranges of two-dimensional non-linear oscillations with periodical Hamiltonian, applied to betatron oscillations in circular partiole accelerators. (Part III).

CERN 57-14 (CERN Reports. Geneva 1957) 44

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