Deformation Quantization*

Or, the phase space formulation of quantum mechanics, and how it really is true that

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[\hat{f},\hat{g}] \to \{f,g\}_{\text{P.B.}}$$

I. PRELIMINARY REMARKS

You've heard that quantizing something canonically consists of working with the corresponding classical system (written in terms of canonical variables x and p), elevating them to operators by 'putting hats on everything' and imposing the canonical commutation relations

$$[\hat{x},\hat{p}] = i\hbar \tag{1}$$

You were then told that by comparing Hamilton's equation in classical mechanics

$$\frac{df}{dt} = \{f,H\}_{\text{P.B.}} + \frac{\partial f}{\partial t} \tag{2}$$

to Heisenberg's equation in quantum mechanics

$$\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f},\hat{H}] + \frac{\partial \hat{f}}{\partial t} \tag{3}$$

where

$$\{f,g\}_{\text{P.B.}} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}, \tag{4}$$

that evidently, quantum mechanics is the $\hbar \to 0$ limit of quantum mechanics. That is,

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[\hat{f},\hat{g}] \to \{f,g\}_{\text{P.B.}}$$

Could this really be true? On the one hand, we have commuting functions on a phase space and on the other hand we have operators on a Hilbert space. There is no meaningful sense in which these things can be limits of each other as they're written above. However there is a precise mapping between operators on a Hilbert space and functions on phase space, where the non-commuting nature of operator products translates into a non-commuting product acting on functions on phase space, allowing us to make this correspondence precise.

II. THE WEYL MAP

Let's start afresh and pause to consider what it means mathematically to 'put hats on everything'. How does one effect this as a mathematical operation? Herman Weyl thought about this, and came up with the following prescription, which has come to be known as

* Supplementary notes for Adv. QM (Kvant 3) 2017; Subodh P. Patil

the Weyl map – it consists of taking any classical function $g(x,p)$ on phase space, and constructing the operator

$$\hat{g}(\hat{x},\hat{p}) = \frac{1}{(2\pi)^2}\int dx\,dp\,d\tau\,d\sigma\,g(x,p)\,e^{i\tau(\hat{p}-p)+i\sigma(\hat{x}-x)} \tag{5}$$

Note that this effects the prescription

$$x \to \hat{x} \tag{6}$$
$$p \to \hat{p} \tag{7}$$

In general, it can be shown that $f(x) \to f(\hat{x})$ and $h(p) \to h(\hat{p})$. But what about more complicated functions of x and p? We observe for example that

$$xp \to \frac{1}{2}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) \tag{8}$$

More generally, any product of x's and p's will return the completely symmetrized product of the operators $\hat{x}$ and $\hat{p}$ under the Weyl map. This is known as 'Weyl ordering'. Recalling now that the Fourier transform of the function $g(x,p)$ is given by

$$\widetilde{g}(\tau,\sigma) = \frac{1}{(2\pi)^2}\int dx\,dp\,g(x,p)\,e^{-i\tau p-i\sigma x} \tag{9}$$

The Weyl map (5) can be written as

$$\hat{g}(\hat{x},\hat{p}) = \int d\tau\,d\sigma\,\widetilde{g}(\tau,\sigma)\widehat{U}(\tau,\sigma) \tag{10}$$

where the operator $\widehat{U}$ is given by[1]

$$\widehat{U}(\tau,\sigma) = e^{i\tau\hat{p}+i\sigma\hat{x}} \tag{11}$$

Using the Campbell-Baker-Hausdorff formula $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$, valid when $[A,B]$ commutes with both A and B, one can show that

$$\widehat{U}(\tau_1,\sigma_1)\widehat{U}(\tau_2,\sigma_2) = \widehat{U}(\tau_1+\tau_2,\sigma_1+\sigma_2)e^{\frac{i\hbar}{2}(\tau_1\sigma_2-\tau_2\sigma_1)} \tag{12}$$

Now, imagine we take the product of two operators $\hat{f}(\hat{x},\hat{p})$ and $\hat{g}(\hat{x},\hat{p})$ constructed from the Weyl map, we find that

$$\begin{aligned}\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p}) &= \int d\tau_1\,d\tau_2\,d\sigma_1\,d\sigma_2\,\widetilde{f}(\tau_1,\sigma_1)\widetilde{g}(\tau_2,\sigma_2)\widehat{U}(\tau_1,\sigma_1)\widehat{U}(\tau_2,\sigma_2) \qquad (13)\\ &= \int d\tau_1\,d\tau_2\,d\sigma_1\,d\sigma_2\,\widetilde{f}(\tau_1,\sigma_1)\widetilde{g}(\tau_2,\sigma_2)\widehat{U}(\tau_1+\tau_2,\sigma_1+\sigma_2)\,e^{\frac{i\hbar}{2}(\tau_1\sigma_2-\tau_2\sigma_1)}\end{aligned}$$

[1] This operator was also used by von Neumann in his proof of the uniqueness (up to unitary equivalence) of the representation of the canonical commutation relations on Hilbert space – with $[x,p] = i\hbar$ defining the so-called Heisenberg algebra [1]. This proof extends to an arbitrary but finite number of canonical pairs, but fails for an infinite number. Therefore in a system with an infinite number of degrees of freedom, not all representations of Heisenberg algebra are unitarily equivalent – a fact that is responsible for spontaneous symmetry breaking [2].

Making the change of variable $\tau = \tau_1 + \tau_2$, $2\bar{\tau} = \tau_1 - \tau_2$ and $\sigma = \sigma_1 + \sigma_2$, $2\bar{\sigma} = \sigma_1 - 2\sigma_2$ the above can be rewritten as

$$\begin{aligned}\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p}) &= \int d\tau\, d\bar{\tau}\, d\sigma\, d\bar{\sigma}\, \widetilde{f}(\tau,\bar{\tau}\sigma,\bar{\sigma})\widetilde{g}(\tau,\bar{\tau},\sigma,\bar{\sigma})\widehat{U}(\tau,\sigma)e^{\frac{i\hbar}{2}(\bar{\tau}\sigma-\bar{\sigma}\tau)} \\ &:= \int d\tau d\sigma \widetilde{f\star g}(\tau,\sigma)\widehat{U}(\tau,\sigma)\end{aligned} \tag{14}$$

where we have defined

$$\widetilde{f\star g}(\tau,\sigma) = \int d\bar{\tau}d\bar{\sigma}\, \widetilde{f}(\tau,\bar{\tau}\sigma,\bar{\sigma})\widetilde{g}(\tau,\bar{\tau},\sigma,\bar{\sigma})e^{\frac{i\hbar}{2}(\bar{\tau}\sigma-\bar{\sigma}\tau)} \tag{15}$$

Comparing the second line of (14) with (10), it seems that the above is the Fourier transform of some function on phase space $f\star g$. What is this function? Inverting the Fourier transform

$$f\star g(x,p) = \int d\tau\, d\sigma\, \widetilde{f\star g}(\tau,\sigma)e^{i\tau p+i\sigma x} \tag{16}$$

and reverting to the variables $\tau_1, \tau_2, \sigma_1, \sigma_2$, we rewrite the above as

$$f\star g(x,p) = \int d\tau_1\, d\tau_2\, d\sigma_1\, d\sigma_2\, e^{i\tau_1 p+i\sigma_1 x}f(\tau_1,\sigma_1)e^{i\tau_2 p+i\sigma_2 x}g(\tau_2,\sigma_2)e^{\frac{i\hbar}{2}(\tau_1\sigma_2-\tau_2\sigma_1)} \tag{17}$$

Now, given that the Fourier transform of the derivatives of f are easily obtained as

$$\begin{aligned}\widetilde{\frac{\partial f}{\partial x}} &= \frac{1}{(2\pi)^2}\int dx\, dp\, \frac{\partial f}{\partial x}(x,p)\, e^{-i\tau p-i\sigma x} \\ &= \frac{1}{(2\pi)^2}\int dx\, dp\, i\sigma f(x,p)\, e^{-i\tau p-i\sigma x} \\ &= i\sigma\widetilde{f}\end{aligned} \tag{18}$$

and similarly for derivatives with respect to p, we see that in general any differential operator acting on phase space $O(\partial_x,\partial_p)$ acting on a function f has the Fourier transform $O(\partial_x,\partial_p)f(x,p) \to O(i\sigma,i\tau)\widetilde{f}$. Therefore it follows from the expression for the Fourier transform of $f\star g$ (17), that it is in fact the Fourier transform of the function

$$f\star g(x,p) = f(x,p)\, e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)}\, g(x,p) \tag{19}$$

where $\overleftarrow{\partial}_x$ denotes a derivative with respect to x acting on whatever is to the left etc. Notice that to first order in $\hbar$, the above is

$$f\star g = f\cdot g + \frac{i\hbar}{2}\{f,g\}_{\text{P.B.}} + O(\hbar^2) \tag{20}$$

Therefore if the Weyl map takes you from the function $f(x,p)$ to $\hat{f}(\hat{x},\hat{p})$ and similarly for $g(x,p)$, then the *operator product* $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p})$ corresponds to the Weyl map of the function

$f \star g$, given by (19)[2]. Given the Weyl map $x \to \hat{x}$, $p \to \hat{p}$, we see that

$$\hat{x}\hat{p} \to x \star p \tag{21}$$

and

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} \to x \star p - p \star x = i\hbar\{x, p\} + O(\hbar^2) = i\hbar + ... \tag{22}$$

More generally,

$$[\hat{f}, \hat{g}] \to f \star g - g \star f = i\hbar\{f, g\}_{\text{P.B.}} + O(\hbar^2) \tag{23}$$

Therefore, via the Weyl map, we see the precise manner in which

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{f}, \hat{g}] \to \{f, g\}_{\text{P.B.}},$$

illustrating the continuous manner in which Quantum Mechanics can be viewed as a one parameter *deformation* of Classical Mechanics, parametrized by $\hbar$, making the correspondence between (2) and (3) mathematically precise. In fact, the Weyl map allows us a third, independent formulation of quantum mechanics (other than canonical quantization and path integral quantization) – the *phase space formulation of quantum mechanics* where wavefunctions are replaced by so-called Wigner functions – see [3] for a nice review.

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[1] J. von Neumann (1931), "Die Eindeutigkeit der Schrödingerschen Operatoren", Mathematische Annalen, Springer Berlin / Heidelberg, 104: 570578; see also `https://en.wikipedia.org/wiki/Stone%E2%80%93von_Neumann_theorem#Uniqueness_of_representation`

[2] F. Strocchi, "Elements Of Quantum Mechanics Of Infinite Systems," Singapore, Singapore: World Scientific (1985) 179 P. (International School For Advanced Studies Lecture Series, 3)

[3] C. K. Zachos, "Deformation quantization: Quantum mechanics lives and works in phase space," Int. J. Mod. Phys. A **17**, 297 (2002) [hep-th/0110114].

[2] The non-commutative product $\star$ is known as the Moyal product, and for the more mathematically minded among you, implements a deformation of the commutative ring of functions on phase space into a non-commutative ring. In other words, the Weyl map represents the algebra of operators on Hilbert space (the Heisenberg algebra) on functions on phase space by promoting the usual multiplicative product $\cdot$ into the (non-commutative) $\star$ product.