An explanation for gravity

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Summary. — This paper is a sequel to reference [1] in which a classical model for reality was presented. Seen from outside, this model world consists entirely of classical mechanical waves in a granular medium possessing a large spherical boundary. Surprisingly, this seemingly barren configuration can be shown to contain quantitative explanations for special relativity, the essence of classical quantum mechanics, the existence of hadrons and leptons and classical electricity including charge and the Coulomb force. An analytic result, not depending on any numerical input, is a spectrum of hadron mass ratios agreeing with experiment within 1% r.m.s. up to about 1.3 GeV. Here it is shown that classical gravity is contained in the same model without requiring any additions, be it in concept or in numerical input. The essence of general relativity is recovered in the form of strain in a medium. But beyond this, the strain's generation by a 'source mass' is explained and related to the fine structure constant and the classical electron radius. Newton's gravitational constant is found within 0.42% of reality.

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1. — Introduction

This paper is a sequel to the theory presented in reference [1] where the unorthodox claim is made that our reality can be described in terms of a classical mechanical, hence visualizable, model. The theory [1] will only be briefly sketched out here. It can, in fact, be condensed into a single sentence:

It looks in considerable detail as if our world consists entirely of waves in a spherical drop of granular jelly.

The statement is meant to be an exact and complete description of the model, as seen by an imagined outside observer, not just a remote metaphor. The details referred to above — that is the physics perceived by the inhabitants of this model universe as far as it has been covered in ref. [1] — are: special relativity, the essence of classical quantum mechanics, the existence of hadrons and leptons (including spectra of 'rest-energy' ratios not depending on any numerical input) and classical electricity including Maxwell's equations, quantized bipolar charge and the Coulomb force. The purpose of this paper is to add classical gravitation — in fact the essence of general relativity — to the list of basic phenomena found in one and the same model.

The first claim contained in the sentence above is the absence from our reality of anything but waves — propagating perturbations in a stationary medium to the exclusion of any ballistic motion. This implies that we live in an entirely kinematic world. Special relativity then follows directly from the assumption that the velocity of light and the spectrum of ‘particle rest-energies’ (or rest-frequencies) form our only reference for space and time. Note that here, as in ref. [1] the outside observer’s point of view is taken whose dynamical entities such as *mass* or *energy* have their Newtonian meaning. The model inhabitants’ point of view with $\hbar = 1$, where the term ‘energy’ stands for a frequency etc., is marked with single quotation marks throughout.

The rest of the model is contained in a description of the wave-propagating medium, called *ether* for obvious reasons although it differs from the 19th-century concept in there being no other material. This ether is supposed to be a classical deformable medium, possessing a mass density ρ, a modulus of rigidity μ, a bulk modulus of compression κ which is very much larger than μ — hence the term *jelly* in the summarizing sentence — and a spherical free boundary of radius R which is the radius of the model universe. These four parameters appear in only two relevant combinations which form the only references of time and distance for the inhabitants of this world. They are:

The propagation velocity c for transverse (shearing) waves given by

(1.1) $$c^2 = \frac{\mu}{\rho}$$

and associated with the velocity of light, and a reference frequency ω_r given by

(1.2) $$(\omega_r R)^2 = \frac{\kappa}{\rho}$$

which sets the scale for a spectrum of longitudinal (compressional) resonances for which the entire model universe forms one (spherical) resonator. These resonances, which are calculated in ref. [1], are associated with the hadron ‘mass’ spectrum and their ratios are found to be in 1% r.m.s. agreement with reality, at least up to about 1.3 GeV. This radically non-local explanation of the hadron mass spectrum (as distinct from all other particle properties) by cavity resonances of the entire universe is clearly the most unorthodox part of the model. But beside the more fundamental justification given in the reference, the idea is directly supported by the fact that the undeniable success of the resulting mass formulae depends entirely on their Neumann-function content: its presence is compelling evidence for the excitation of a spherical object *from its inside*.

Microscopic processes are supposed to be governed by the granularity of the medium which is to have an ordered molecular structure with molecular radius r_o, say. ‘Particles’ in this model are single-period wave-packets centred in *dislocations* in this medium. Indeed, a dislocation is the only logically tenable entity which *is* a wave but behaves like a particle. Note that the molecules invoked here are stationary and all identical; it is the perturbation of their order that propagates.

A permanent dislocation — an excess or deficit molecule — is associated with electric charge which, therefore, is quantized in multiples of one molecular volume

(1.3) $$V_o = \frac{4\pi}{3} r_o^3$$

counted in opposite polarities for excesses and holes. The electric field, then, is the ether displacement from rest position created by (and decreasing with inverse square distance from) such a dislocation.

Finally, the medium is supposed to possess the plausible property of losing its rigidity if it is distorted beyond a certain shearing angle. This is called *disruption angle* hereafter as well as in ref. [1] where it has been made the basis for a consistent and quantitative explanation for the Coulomb force, the electron rest 'energy' and the associated phenomenon of a classical electron radius. It follows unambiguously from these mutually consistent associations that the disruption shear angle ϑ and the molecular radius r_o are known to us and given by

$$\vartheta = \left(\frac{\alpha}{4}\right)^2 \tag{1.4}$$

$$r_o^3 = \vartheta \left(\frac{r_e}{2}\right)^3 \tag{1.5}$$

where α = 1/137.04 is the fine structure constant and $r_e = 2.8179 \times 10^{-15}\mathrm{m}$ is the classical electron radius. The shear angle ϑ refers to the spherical strain surrounding a point-like dislocation. The present theory will be concerned with planar shear for which the disruption angle, $\varepsilon_d = \left(\partial s_x / \partial y + \partial s_y / \partial x\right)$, amounts to two-thirds of ϑ. It thus follows from the theory of ref. [1] that

$$\varepsilon_d = \frac{2}{3}\vartheta = \frac{\alpha^2}{24} = 2.218\,806 \times 10^{-6}\mathrm{rad} \tag{1.6}$$

$$r_o = \vartheta^{\frac{1}{3}} \frac{r_e}{2} = 2.103\,643 \times 10^{-17}\mathrm{m} \tag{1.7}$$

The two parameters — and hence the two 'electrical' constants of nature α and r_e — form the only numerical input for this explanation of gravity.

For what has been discussed so far the entities μ, ε_d, r_o might have retained an abstract status. They can, however, be perfectly described as resulting from the following model which was already presented in [1].

The ether molecules of radius r_o are taken as frictionless spheres which incorporate the bulk modulus κ. Since the associated compressional velocity $\sqrt{\kappa/\rho}$ is supposed to resonate with the size of the entire model universe the molcular spheres can be considered as incompressible over any distance much smaller than this universe; they will be so considered for the rest of this paper. The assembly of hard spherical grains is supposed to be held in nearly close packing — every sphere contacting 12 others — by an external pressure p_μ. The presence of distributed dislocations, which forms an essential part of the model anyhow, will create a small distibuted deformation, by an (average) shear-angle ε_d, say. This *prestrain*, as it will be called hereafter, prevents the structure from locking solid. The deformability thus gained is, however, limited to dynamic shear angles $\Theta < \varepsilon_d$, beyond which the freedom of *reversible* deformation is exhausted and disruption occurs.

Even below the disruption limit ε_d, any deformation by a shear-angle Θ must be expected to create a small second-order volume inflation δV which can be written in the form

$$\frac{\delta V}{V} = g\Theta^2 \tag{1.8}$$

to first approximation. This will be confirmed and the coefficient g calculated in Section 6. Creating the excess volume δV against the packing pressure p_μ requires energy. It, thus, creates an equivalent rigidity

$$\mu = 2gp_\mu \quad . \tag{1.9}$$

Since the geometrical coefficient g can be calculated, the new parameter p_μ merely replaces μ.

This model of nearly close-packed spheres offers immediately plausible explanations for the independence of μ from κ and for the smallness of $\varepsilon_d = \alpha^2/24$. It was already introduced in ref. [1] and will now be explored further.

2. — Outline of a model for gravity

In order to demonstrate classical gravitation in the terms of this model it must be shown that *any* wave packet — called test wave or test 'particle' — propagating (with v << c) through the elastic medium just described at a distance r from another wave-packet — called source 'particle' — suffers a centripetal acceleration of its group velocity v given by

$$\dot{v}_r = -\frac{Gh\omega}{c^2 r^2} \quad . \tag{2.1}$$

Here ω is the carrier frequency of the source wave — thus $h\omega/c^2$ its 'mass' — and G is Newton's constant of gravitation, the accepted experimental value being $6.6726\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$ [2]. Moreover, the effect must be additive so that an ensemble of source particles of 'masses' $h\omega_i/c^2$ at stationary positions $\mathbf{r}_i$ in a co-moving co-ordinate system generate a universal group acceleration at position $\mathbf{r}$ as given by

$$\dot{\mathbf{v}} = -\frac{Gh}{c^2}\sum_i \omega_i \frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3} \quad . \tag{2.2}$$

The frequencies ω_i are to be the actual frequencies, including 'binding energies' which are due to mutual refraction in this model.

It turns out that the solution presented here does include relativistic motion of the test wave, where $\dot{\mathbf{v}}$ is to be replaced by the gradient of a frequency potential $\omega(r)$. This will be briefly discussed in an Appendix, although the main purpose of this paper is the demonstration of a fundamental mechanism which exposes a *source* of gravity, as distinct from describing its effect in terms of distorted space.

The source of gravity proposed here is the second-order oscillatory volume perturbation δV — eq. (1.8) with $\Theta = \hat{\Theta}\cos\omega t$ — which every transverse wave and, thus every 'particle', necessarily produces as a side effect in the granular medium. The model ether described above makes such an effect inescapable. The idea turns out quantitatively successful although problems seem to appear at every step of its analysis. In this Section, these problems are outlined together with the natural solutions they all find. Quantitative analysis then follows in the remainder of the paper.

First of all it is evident that gravity is the incoherent sum of its sources in spite of its proportionality to the sum of source frequencies rather than their squares. The volume perturbation (1.8), being proportional to $\cos^2\omega t$, does produce a constant source-volume expansion but this static effect merely adds to the prestrain and does not produce any observable group acceleration.

A non-oscillatory *dynamic* effect does, however, exist. It is due to the fact that the oscillatory part of the volume perturbation δV — (1.8) with $\theta^2 \propto (1+\cos 2\omega t)$ — necessarily forms a monopole source of compressional radial oscillation in which the oscillatory radial displacement and the associated pressure gradient (mass density times acceleration) are coherently in phase. This creates a time-independent *wave-pressure* at the boundary of the volume perturbation where the radial waves have their origin.

The reaction to this inner boundary pressure is a field of elastic stress which decreases with inverse distance in the medium. In undisturbed regions this stress is static, the stress lines being closed in themselves. But every test wave embedded in the stressed medium carries its own volume perturbation δV_2 at its centre. The stress gradient to which it is exposed imparts a net centripetal body-force and concomitant acceleration on the perturbed volume and, hence, on the test-wave centre. The magnitude of δV_2 cancels out since it determines the force as well as the mass defect to be accelerated. The acceleration is proportional to δV^2 (and thus to Θ^4) in the source wave and it decreased with r^{-2}. It follows that *every* (non-relativistic) wave propagating in the neighbourhood of the source 'mass' suffers a centripetal group acceleration equal to this local acceleration of the medium.

The effect is proportional to ω_i^2, not ω_i, as indeed it must be, since it is incoherent. The problem disappears when the total volume of a source 'particle' is considered. This, in this model, is defined as a single-period wave-packet whose centre is a propagating dislocation in the granular medium. The transverse size is the size of the granularity (associated with the molecular diameter $2r_o$) as the smallest dimension over which the medium is deformable. In the direction of propagation, on the other hand, the r.m.s. length of the wave-packet is the wavelength c/ω_i, thus cancelling one factor ω_i (the absence of any special relativistic factor in this context is discussed in Section 5). Only the non-linear core, the dislocation itself, has been considered thus far but integrating the contribution due to the linear part of the wave — the halo so to speak — only adds a factor two.

Thus a universal and inescapable centripetal group acceleration of any test wave has been found in the model. It is proportional to $\sum_i \hat{\hat{\Theta}}_i^4 \omega_i / r_i^2$ where $\hat{\hat{\Theta}}_i$ is the absolute (spatial and temporal) peak of dynamic shear angle in a source wave-packet. As a dependence on individually different oscillation amplitudes $\hat{\hat{\Theta}}_i$ is at variance with reality the seemingly *ad-hoc* assumption of a standardized peak shear angle — the same for every wave-packet in the universe — is required. Such a standardization imposes itself quite naturally, however, in the form of the disruption angle ε_d at which the medium loses its rigidity and which has been shown in [1] to be essentially identical with the square of the fine-structure constant α [*cf.* (1.6)]. It is clearly equal to the maximum shear-angle amplitude possible for a linear transverse wave in this medium and all that is required is the assumption that all wave-packets — thus all 'particles' in this universe — do reach the limiting amplitude by virtue of their creation (namely by non-linear disruption of molecular order and, thus, by overstraining the wave-propagating medium or 'vacuum').

At this point (end Section 5) eq. (2.2) is recovered without a free parameter left. What remains is a seemingly trivial problem of classical geometry, namely the calculation of the coefficient g relating volume inflation δV to planar shearing Θ in a prestrained ensemble of close-packed spheres. If one were to follow one's intuitive expectation that g must be of order unity (even specifically $\frac{1}{2}$ as in a simple cubic lattice but with opposite sign) one would find the equivalent of Newton's constant of gravitation about 12 orders of magnitude larger than reality. It turns out, however (Section 6), that the minimum value of g — and thus the one that nature will choose — is much smaller, namely related to the prestrain and equal to $2\varepsilon_d/9$.

With this result, the constant of gravitation G is numerically recovered — from contributing factors spanning 127 orders of magnitude — at 0.4% from reality. The universality of the effect and its attribution to a distortion of the ether — thus of 'space' in orthodox parlance — rather than to any feature of the test wave, reproduces the fundamental essence of general relativity.

The next five sections contain the quantitative analysis of what has been outlined here. Much of it, unavoidably, is concerned with elementary if tedious geometrical considerations aimed at obtaining as accurate a determination of G as possible without having to resort to computational studies of perturbed close-packing and its statistics. A connection with general relativity beyond Newtonian gravitation is made in an Appendix where the Schwarzschild metric is readily recovered.

3. — Prestrained closed packing

The medium considered throughout this theory is an assembly of hard molecular spheres of radius r_o compressed to very near maximum packing density. Every sphere is in contact — or very nearly so— with 12 neighbours (*12-connection*). The volume occupied by one sphere is given by

$$V_1 = \frac{3\sqrt{2}}{\pi} V_o \tag{3.1}$$

where V_o is the hard sphere volume (1.3) and $3\sqrt{2}/\pi$ is the inverse of the packing efficiency. The configuration, therefore, is either face-centred cubic (fcc), hexagonal or a mixture of the two. That this, rather than any random configuration, is indeed the closest packing appears to be finally established [3]. To fix ideas, an fcc lattice will be assumed throughout, although it seems probable that a hexagonal configuration gives the same results.

A fundamental feature of the model is the presence of dislocations. The molecular spheres are assumed frictionless and incompressible over any distance much smaller than the model-universe. Therefore, the total number of dislocations in a very large volume will create a uniformly distributed perturbation of 12-connection by opening a fraction of intermolecular contacts. This will be called *prestrain*.

Outside the immediate neighbourhood of a dislocation the prestrain must be isotropic which means that its creation does not produce any displacement at large distance other than uniform volume inflation. A possible (and possibly unique) configuration can be described as follows.

In face-orthogonal projection the face-centred cube is seen to be composed of eight smaller cubes of sides $\sqrt{2}r_o$ — called *subcubes* here. Each subcube is defined by its face diagonals which are identical with the six edges of an inscribed tetrahedron, each edge of length $2r_o$ being the contact line joining the centres of neighbouring spheres. Isotropic prestrain is created by subjecting each subcube to planar shear angles ε_d (the deviation from 90° corner-angle in a cube face) in all three of its planes. This opens half of all contacts in total. There are two ways of doing this, namely by opening intersphere contacts either belonging to one common triangle or to one common vertex of an inscribed tetrahedron. Combining both types in alternate positions and alternating orientations makes them nest into a complete (distorted) fcc in an apparently unique way. (No proof of uniqueness can be offered but no other pattern conserving the original lattice periodicity has been found. An analogous pattern at twice the original period is, however, obtained by applying the procedure just described to complete fcc's rather than to subcubes.) Random prestrain may be possible but appears unlikely. In any case, the only thing required here is a proof of existence of isotropic prestrain in which half of the 12 contacts per sphere, on average, are opened by a very small amount.

The prestrain is a property of undisturbed ether — or '*vacuum*' — in this model. It permits subsequent dynamic shear by an angle $\Theta < \varepsilon_d$ as part of transverse-wave propagation. At $\Theta = \varepsilon_d$ the medium loses its rigidity by disruption of molecular order.

4. — A source element of gravity

A coordinate system fixed in the medium, but centred in the instantaneous position of the packet of transverse waves which forms the individual source 'particle' under consideration here, will be employed. The carrier-frequency component of shear ($\Theta = \partial s_x / \partial z$ for propagation in z direction and polarization in x direction) at the source may be taken as

(4.1) $$\Theta(t) = \hat{\Theta} \cos \omega t \quad .$$

The smallest deformable element in the prestrained granular medium is the volume V_1 of (3.1) occupied by one molecular sphere in the (nearly) close-packed lattice. The prestrain has opened half of the 12 possible contacts of each sphere with its neighbours. The dynamic shear (4.1) opens additional ones and tends to close others. It creates a second-order volume perturbation of the form (1.8) in which the denominator is to be taken as V_1. The resulting oscillatory volume perturbation

(4.2) $$\delta V = V_1 g \hat{\Theta}^2 \cos^2 \omega t$$

will now be taken as a spherical monopole source of radial oscillation, of volume V_1 and concomitant spherical radius r_1. Outside this volume (and *a fortiori* at $r >> r_o$) the medium is treated as a continuum.

At the surface of this equivalent continuum-source a radial displacement $s_1(t)$, a radial velocity $\dot{s}_1(t)$, a local radial acceleration $\ddot{s}_1 = \partial \dot{s} / \partial t$ and a volume density of average kinetic energy w_1 occur. These are, thus, necessary byproducts of every transverse wave ('particle') in this model and are given by

(4.3) $$s_1(t) = \frac{\delta V}{4\pi r_1^2} = \frac{1}{6} g \hat{\Theta}^2 r_1 (1 + \cos 2\omega t)$$

(4.4) $$\dot{s}_1(t) = -\frac{1}{3} \omega g \hat{\Theta}^2 r_1 \sin 2\omega t$$

(4.5) $$\ddot{s}(t) = -\frac{2}{3} \omega^2 g \hat{\Theta}^2 r_1 \cos 2\omega t$$

(4.6) $$w_1 = \rho \dot{s}_1^2 = \frac{1}{18} \rho \omega^2 g^2 \hat{\Theta}^4 r_1^2$$

respectively, where ρ is the bulk mass density of the medium and r_1 is related to the molecular radius r_o by

$$r_1^3 = \frac{3\sqrt{2}}{\pi} r_0^3 \tag{4.7}$$

The velocity and local acceleration are purely oscillatory. However, since $-\rho\ddot{s}$ defines an oscillatory pressure gradient which is in phase with the oscillatory part of the displacement, a steady average pressure is generated. It is identical with the hydrostatic pressure which is present in every wave and equal to the average density of kinetic energy. At the boundary radius r_1, where the radial wave is reflected, this *wave pressure* amounts to

$$2w_1 = \frac{1}{9}\rho\omega^2 g^2 \hat{\Theta}^4 r_1^2 \quad . \tag{4.8}$$

The reaction to this boundary pressure must be a stress field in the medium which matches (4.8) at r_1 and vanishes at large distance. A solution is offered by a field of constant and positive radial displacement — the same for all radii — which will be called $\delta a/2$ for later convenience. Since this radially outward displacement reduces the packing density, it does not couple directly to central intermolecular forces — and thus to the modulus of compression κ — but only to the rigidity μ. The only finite elements of the concomitant strain tensor are

$$\varepsilon_{\Theta\Theta} = \varepsilon\phi\phi = \frac{\delta a}{2r} \quad . \tag{4.9}$$

The finite divergence tends to create a positive gradient of hydrostatic pressure $\mu\ \delta a/r^2$, but the radially inward acceleration potentially associated with this blocks itself at vanishing inward displacement by central forces between incompressible molecules. There remains a tangential tension

$$\sigma_T = \sigma_{\Theta\Theta} = \sigma_{\phi\phi} = \mu\frac{\delta a}{r} \tag{4.10}$$

surrounding the source. This matches the boundary pressure (4.8) for

$$\delta a = \frac{1}{9}\frac{\omega^2}{c^2} g^2 \hat{\Theta}^4 r_1^3 = \frac{\sqrt{2}}{3\pi}\frac{\omega^2}{c^2} g^2 \hat{\Theta}^4 r_o^3 \tag{4.11}$$

where, generally, $c^2 = \mu/\rho$ is the velocity of light in this model.

It may be noted in passing that the second-order radial oscillation (4.3) to (4.5) is necessarily a compressional one, propagating with the compressional velocity $(\kappa/\rho)^{\frac{1}{2}}$. It is reflected at the outer radius R of this model universe and part of the cavity resonances (1.2) associated (and found in agreement) with ‘particle rest energies’. This precludes any radiation loss to the primary transverse wave $\Theta(\mathbf{r},t)$. On the other hand, the stress field (4.10) caused by the non-oscillatory pressure (4.8) couples to the rigidity μ only. Its variations, therefore (hence those of ‘gravitation’ in this model) propagate with the velocity of light.

In the absence of any perturbation outside the source, the situation is static, the stress lines forming closed circles. But every test wave immersed in the stress field (4.10) carries its own excess volume δV_2 (per participating molecule) at its centre. Creating this volume expansion within the tension σ_T releases the energy $\sigma_T \delta V_2$ so that the gradient of tension creates the radial body-force

$$F_2 = \delta V_2 \frac{\partial \sigma_T}{\partial r} = -\mu\frac{\delta a}{r^2}\delta V_2 \quad . \tag{4.12}$$

Dividing this by $\rho\delta V_2$ (the return mass), substituting (4.11) for δa and making use of $\mu/\rho = c^2$ again, yields a universal centripetal acceleration

$$\delta\ddot{s} = -\frac{1}{9}\omega^2 g^2 \hat{\Theta}^4 \frac{r_1^3}{r^2} = -\frac{\sqrt{2}}{3\pi}\omega^2 g^2 \hat{\Theta}^4 \frac{r_o^3}{r^2} \qquad (4.13)$$

of every test-wave centre at distance r from the source.

The expression (4.13) is found to be identical with the field of acceleration which the inner-boundary pressure (4.8) would impart on the bulk of the medium if it were a liquid and had a sink at its centre. Instead, the material acceleration (4.13) is limited to the microscopic volume perturbation at the centre of a test wave in an otherwise static but stressed medium. The tension (4.10) pulls at the void δV_2 at right angles to the axis formed by the two wave centres. But the gradient of the tension, being radial, opens the void where it faces the source and closes it at its back, hence moving it towards the source. Thus, although the actual microscopic motion of bordering molecules following the stress (4.10) is tangential, the boundary wall of the volume perturbation δV_2 (formed by molecules of ever-changing identity) moves towards the source and a deficit of mass $-\rho\delta V_2$ moves with it.

The essence of an explanation for gravity is already clearly visible: a stationary distortion of ‘space’ (the propagating medium here) surrounds every (source) wave centre. The gradient of the associated stress, which decreases with the square of inverse distance, imparts on every test-wave centre a radially inward acceleration, with the consequence that any test wave propagating in this ‘space’ suffers an inescapable group acceleration towards the source — and *vice versa*.

The notations $\delta\ddot{s}$ and δa indicate that these entities are the contributions from but one source molecule at a time. Many such contributions must be added to form the effect of a complete source wave-packet. This is the subject of the next section.

5. — The complete source particle

A ‘particle’ in this model — as viewed from outside — is a single-period Gaussian wave-packet centred in a propagating dislocation. Accordingly, the dimensions of its core transverse to propagation are the ones of the granularity and, therefore, of the source element analysed in the preceding Section. In the direction of propagation (in z-direction, say), the core is a string of such source elements forming a travelling transverse wave as given by

$$\Theta(z,t) = \hat{\hat{\Theta}} e^{-\frac{\omega^2}{2c^2}(z-\beta ct)^2} \cos\omega\left(t - \beta\frac{z}{c}\right) \qquad (5.1)$$

for a dipole mode. Here, $\Theta(z,t)$ is the instantaneous shear angle ($\partial s_x/\partial z$ for polarization in x direction), $\hat{\hat{\Theta}}$ its absolute (spatial and temporal) peak, c/β the phase velocity of the carrier wave and βc the group velocity of the Gaussian wave-envelope.

This is the outside observer’s (or model constructor’s) point of view and it is in agreement with the single-period definition of a ‘particle’ that the rms length of the corresponding wave packet is one vacuum wavelength over 2π, thus c/ω in (5.1). The length in this external view of a Newtonian-Galilean model is *not* affected by any factor $\sqrt{1-\beta^2}$, notwithstanding the fact that co-moving model inhabitants see the ‘particle’ at ‘rest energy’ ω_o and zero ‘momentum’, so that the bandwidth theorem (or ‘uncertainty relation’ in orthodox language) keeps them from localizing its centre.

Each granular source element along z contributes its centripetal acceleration (4.13) incoherently. For propagation orthogonal to fcc faces these source elements follow each other in zigzag at distance $\Delta z = \sqrt{2} r_o$. This standard orientation will be assumed from here on, and isotropy taken for granted, although skew propagation remains to be checked. The total effect of one wave packet at $r \gg r_o$ is, therefore, obtained by taking the fourth power of (5.1), substituting it into (4.13), integrating over $dz/\sqrt{2} r_o$ and taking the average by dropping oscillatory terms. Since t = 0 can be chosen to coincide with the peak of the envelope, this amounts to forming

$$\ddot{s}(r) = -\frac{r_o^2}{3\pi r^2}\omega^2 g^2 \hat{\hat{\Theta}}^4 \int_{-\infty}^{\infty} e^{-2\frac{\omega^2 z^2}{c^2}} \cos^4\left(\beta \frac{z}{c} - \psi\right) dz \qquad (5.2)$$

and averaging over ψ, which is the phase slip between the envelope and the carrier. With this, the integral amounts to $\left(3\sqrt{\pi}/8\sqrt{2}\right) c/\omega$. Finally, as has been discussed in Section 2 the absolute peak shear angle $\hat{\hat{\Theta}}$ is identified with the disruption angle ε_d . The result is

$$\ddot{\bar{s}}(r) = -\frac{1}{8\sqrt{2\pi}} c\,\omega\, g^2 \varepsilon_d^4 \frac{r_o^2}{r^2} \quad . \qquad (5.3)$$

It remains an open question (in this theory as well as experimentally) as to what happens to a gravitational source at multi-GeV energies when c/ω approaches r_o, although the conjecture may be permitted that (5.3) remains valid on *average*.

So far, only the granular core of the wave has been considered. The surrounding transverse wave Θ (r*, φ*, z, t) in cylindrical coordinates r*, φ*, z, is a cylindrical dipole mode excited by a cylindrical source of radius $r_o^* \sim 2r_o$. Thus, for given z and t, Θ(r*) decreases with $\left(r_o^*/r^*\right)^2$, namely with the small-argument asymtote of the cylindrical Neumann function. At cylinder radius r* the shear amplitude is given by

$$\Theta\left(r^*\right) = \Theta\left(r_o^*\right)\left(\frac{r_o^*}{r^*}\right)^2 \quad . \qquad (5.4)$$

Again, the contribution to Θ^4 of all molecular sources in this halo volume outside r_o^* must be added incoherently. Along the z-axis each central molecule in the zigzag string occupies $\Delta z = \sqrt{2} r_o$. In the medium at large each molecule occupies a volume $4\sqrt{2} r_o^3$. Therefore, the number of halo sources per central source, weighted with the fourth power of amplitude, is given by

$$\nu = \frac{1}{4 r_o^2} r_o^{*8} 2\pi \int_{r_o^*}^{\infty} \frac{r^* d r^*}{r^{*8}} = \frac{\pi}{12}\left(\frac{r_o^*}{r_o}\right)^2 \qquad (5.5)$$

with the consequence that the right-hand side of (5.3) must be multiplied with (1 + ν) to add the halo contribution.

The difficulty here is to determine r_o^*/r_o It appears, nevertheless, that $r_o^* \approx 2r_o$ is a good estimate, since this is where the centres of contacting (or nearly contacting) neighbours to the central molecule are situated. In the reference orientation chosen there are four such neighbours within $\Delta z = \sqrt{2} r_o$ (at 90° from the axis and 45° from each other). They constitute the closest objects which are capable of independent transverse motion and may, therefore, be expected to move on average in accordance with continuum kinematics.

A more precise definition of r_o^*, which differs from the above estimate by only $\sqrt{3/\pi}$, can be derived from considering the ratio of transverse-wave kinetic energies outside the core and inside. In a continuum, excited by a bodily oscillating central rod of the same density, this ratio is one. Here, if the four spheres mentioned above are counted as being half within r_o^*, the equivalent of three spheres is found within r_o^* and $\Delta z = \sqrt{2} r_o$. Therefore, the ratio of kinetic energies outside r_o^* and inside must be taken as

(5.6) $$\frac{\pi}{3\sqrt{2}}\sqrt{2}r_o 2\pi r_o^{*4}\int_{r_o^*}^{\infty}\frac{r^* dr^*}{r^{*4}}\left[3\frac{4\pi}{3}r_o^3\right]^{-1} = \frac{\pi}{12}\left(\frac{r_o^*}{r_o}\right)^2$$

where $\pi/3\sqrt{2}$ is the packing factor outside r_o^*. Setting this expression to unity, so as to match the continuum situation, gives $r_o^* = 1.9544 r_o$ and $\nu = 1$ exactly from (5.5). It implies an exact factor two to be applied to (5.3) resulting in

(5.7) $$\ddot{\bar{s}} = -\frac{1}{4\sqrt{2\pi}} c\omega g^2 \varepsilon_d^4 \frac{r_o^2}{r^2}$$

So far, only dipole oscillations have been considered while, in the frame of this model [1], baryons are associated with modes of higher orders and nucleons, in particular, with quadrupole modes. However, individual ether molecules can only carry out dipole oscillations and their contributions to $\ddot{s}$ add up incoherently. Moreover, the fields of displacements and shear angles considered here can all be derived from potentials — ϕ_l, say, for the order _ — for which, generally

(5.8) $$\phi_{l+1} = \nabla\phi_l \cdot \delta\mathbf{r}$$

where the magnitude of the small vector $\delta\mathbf{r}$ is $\sqrt{2}r_o$, namely the smallest projected intermolecular distance transverse to propagation. Equation (5.8) describes the superposition of two fields of order _ and opposite signs whose axis of propagation are parallel at the small distance δr.

The quadrupole mode, in particular, consists of a pair of opposite-sign and slightly offset dipole modes. The microscopic details are irrelevant here(1). In each of the two dipole waves, running parallel at $\sqrt{2}r_o$ distance, the distribution of shear angles has the z-dependence (5.1) and the absolute peak angle equals the disruption-shear ε_d as before. The core now contains the equivalent of five spheres within $\Delta z = \sqrt{2}r_o$ (two central ones and six halves). This implies a multiplication of (5.6) by a factor 3/5. But the same factor results from the integration in (5.5) where r^{*-8} is to be replaced by r^{*-12} since the quadrupole amplitude scales with r^{*-3}, thus Θ^4 with r^{*-12}. The net result is $(1+\nu) = 2$ as before and (5.7) remains unchanged. It appears that the argument can be extended to arbitrary orders _ (while it is also true that, in this model, modes with _ > 2 correspond to hyperons whose properties as sources of gravitation remain safely unobservable).

Equation (5.7) is the net result of all arguments thus far. It describes a local and instantaneous centripetal acceleration of the volume perturbation at the centre of every test wave at distance r from a source 'mass' $h\omega/c^2$. The model is essentially complete at this point since it seems evident (and will be confirmed in the Appendix) that any such test wave

(1) It is the author's opinion, noted in passing here, that a detailed study of the granular-core motion in these modes might reveal the quark structure.

(with $v_g << c$) will suffer a group acceleration dv_g/dt eqal to the kinematic acceleration (5.7) of the medium at its centre. Note that the local acceleration $\ddot{\bar{s}}$ does not, in general, follow the test wave which may have arbitrary velocity and direction. The restriction to $v_g << c$ will be removed in the Appendix.

Clearly the ether strain due to a complete source particle results from its molecular contributions, given by δa of (4.11), by the same process of integration as described above — thus, in final analysis, by multiplication with a factor $\left(3\sqrt{\pi}/8\right)\left(c/\omega r_o\right) = 0.6647 D/r_o$. The corresponding constant radial displacement amounts to

(5.9) $$\frac{a}{2} = \frac{1}{8\sqrt{2\pi}}\frac{\omega}{c} g^2 \varepsilon_d^4 r_o^2 \quad .$$

It corresponds to one quarter of the *Schwarzschild radius* with which it will be found to agree numerically.

Equations (5.7) and (5.9) contain no free parameters since ε_d and r_o only express the fine structure constant and the classical electron radius via (1.6) and (1.7). The geometrical parameter g remains to be determined, however, and this will be done in the next section.

6. — The coefficient of volume expansion

An unperturbed ensemble of close-packed hard spheres of radius r_o consists of tetrahedra, their edges of length $2r_o$ connecting the centres of contacting spheres. A uniform prestrain, due to the presence of distributed dislocations, is supposed to have opened a fraction of the intersphere contacts so as to create reversible deformability up to a maximum planar shear angle ε_d (the disruption angle) while maintaining macroscopic isotropy. An external pressure p_μ imposes minimum volume. The pressure creates (and is essentially identical with) rigidity μ and, thus, the possibility of transverse wave propagation with maximum velocity $c = \sqrt{\mu/\rho}$.

It seems safe to assume that the prestrain is uniformly distributed over all tetrahedra and that only one contact per tetrahedron is opened for each one of the three orthogonal pre-shears forming the complete prestrain — thus, half the total number of contacts — since this minimizes the volume inflation. Only these features are really required but the particular configuration of an isotropically prestrained fcc presented in Section 3. — will be assumed here. It has the required properties. It may or may not be unique.

Thus, each tetrahedron has three of its six edges (obviously shared with its neighbours) increased from $2r_o$ to

(6.1) $$s_o = \left(1 + \alpha_o\right) 2r_o$$

say. The pre-opened contacts are free of force — or else they would collapse — and the prestrained assembly, although deformable, remains incompressible for the same reason. It follows that it is the skeleton of loaded contacts, corresponding to the non-extended sides of all tetrahedra, which remains incompressible and retains the ability to generate radial acceleration, not the empty space in between. If, therefore, this skeleton is subjected to coherent distortion by the planar shear angle Θ associated with a transverse wave, this process can only lead to a net increase of the total number of contacts opened. As this works against the coherence pressure p_μ, it creates rigidity.

The packing pressure is a continuum concept. To match it to a set of discrete forces, f_c per newly broken contact, the total force $F_c = 2r_o^2 p_\mu$ on a pair of opposite faces of the subcube with edges $\sqrt{2}r_o$ introduced in Section 3. — (one eighth of a complete fcc) must be divided among the skew contact-lines joining these faces. There are four such contacts but they are all shared with neighbours and half of them are force-free due to the prestrain, leaving one loaded contact per subcube. Moreover, the contact lines run at 45° with respect to the cube faces. The force per contact, therefore, is given by

(6.2) $$f_c = \sqrt{8}r_o^2 p_\mu \quad .$$

Opening a contact by an amount $2\alpha_1 r_o$,say, requires the work $\sqrt{32}\,\alpha_1 r_o^3 p_\mu$. Equating this to a continuum work $p_\mu \delta V$ defines a continuum volume expansion given by

(6.3) $$\delta V = \sqrt{32}\,\alpha_1 r_o^3 \quad .$$

Dividing this by the volume of the subcube, $\sqrt{8}\,r_o^3$, gives the fractional volume perturbation in terms of the contact stroke $2\alpha_1 r_o$, and thus

(6.4) $$g\Theta^2 = 2\alpha_1 \quad .$$

Only the volume inflation defined in this way — namely working against a pre-established closure force — contributes to the inflation of the skeleton of loaded contacts as well as to finite rigidity via (1.8) and (1.9). The coefficient α_o, which refers to pre-opened and unloaded contacts, makes no such contributions. On the other hand, since only pre-opened contacts can be closed as well as opened further, it is α_o which is affected in the first place (and to first order) by the dynamic shearing angle Θ.

The minimum α_o compatible with a given shear angle Θ occurs in a plane which contains a pre-extended edge s_o of the tetrahedron and is orthogonal to the opposite edge. In this projection, in which the tetrahedron is viewed along the edge opposite to s_o, it is seen as an isosceles triangle with two fixed sides $r_o\sqrt{3}$ and one variable side $s_o = 2r_o(1+\alpha_o)$. The height h from the vertex to the midpoint of s_o equals $r_o\left[3-(1+\alpha_o)^2\right]^{\frac{1}{2}}$. The total shear-angle

(6.5) $$\varepsilon = \varepsilon_d - \Theta$$

in this plane, associated with changing the broken-contact line s_o, is the deviation from 90° between the diagonals of a perturbed square whose sides are $\sqrt{2}h$ and s_o. This is, therefore, given by

(6.6) $$\tan\left(45° + \frac{\varepsilon}{2}\right) = \sqrt{2}\,\frac{1+\alpha_o}{\left(3-(1+\alpha_o)^2\right)^{\frac{1}{2}}} \quad .$$

Inverting the function and developing $\alpha_o(\varepsilon)$ in powers of ε results in

(6.7) $$\alpha_o = \tfrac{2}{3}\varepsilon - \tfrac{1}{27}\varepsilon^3\,\mathrm{K} =$$
$$= \left(\frac{2}{3}\varepsilon_d - \frac{1}{27}\varepsilon_d^3\right) - \left(\frac{2}{3} - \frac{1}{9}\varepsilon_d^2\right)\Theta - \frac{1}{9}\varepsilon_d\Theta^2 + \frac{1}{27}\Theta^3\,\mathrm{K} \quad .$$

The crucial and unexpected result is that the second order in ε is zero. The first term in the second part of (6.7) is the prestrain. The terms with Θ and Θ^3 cancel out when tetrahedra of different orientations in space are subjected to the same global shearing Θ. What remains is the Θ^2 term.

The dynamic shearing by an angle Θ acts at the skeleton of loaded contacts from its outside, which means that the quantity α_o in (6.7) refers to pre-opened (force-free) contacts. Only in this way can a first-order volume increase be avoided since originally closed contacts can only be opened but not closed further. Thus the remaining second-order closure of pre-opened contacts indicated by the negative Θ^2-term in (6.7) means stretching the skeleton from its outside. As the spheres are rigid, the stretch implies the opening of additional contacts (against the closure force f_c) to the equivalent of the Θ^2-term of (6.7).

Thus, the sign reverses from α_o of (6.7) to the quantity α_1 in (6.4) which is therefore given, on average, by

(6.8) $$\overline{\alpha}_1 = \frac{\varepsilon_d}{9}\Theta^2$$

so that, in final analysis, the minimum value of g, and hence the one the system chooses, is given, via (6.4), by

(6.9) $$g = \frac{2}{9}\varepsilon_d \ .$$

7. — Newton's gravitational constant

Equation (6.9) is the keystone for this model of gravity. The fact that the coefficient g turns out to be $2\varepsilon_d/9$ rather than of order unity (as a finite second-order term in the middle part of (6.7) would have imposed) saves the result from emerging 12 orders of magnitude larger than reality!

As it turns out, substitution of (6.9) into (5.7) changes the dependence on ε_d from fourth to sixth power and yields the final result of

(7.1) $$\overline{\ddot{s}} = -\frac{1}{81\sqrt{2\pi}}c\omega\varepsilon_d^6\frac{r_o^2}{r^2} \ .$$

It describes a local centripetal acceleration of the propagating medium — and thus of 'space' — at the centre of every test wave towards any source 'particle' of 'energy' ω. Every non-relativistic test wave in the medium inescapably experiences a group acceleration $\dot{v}$ equal to $\overline{\ddot{s}}$. Moreover, as the factor $c\omega$ really stands for $\omega^2 D$, the effects of multiple source particles are incoherently additive. Equation (2.2) is fully recovered, therefore. It may be written in the form

(7.2) $$\dot{\mathbf{v}} = \overline{\ddot{\mathbf{s}}} = -G\sum_i m_i \frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}$$

where $m_i = h\omega_i/c^2$ is the 'mass' of a source 'particle' and

(7.3) $$G = \frac{1}{81\sqrt{2\pi}}\frac{c^3}{h}r_o^2\,\varepsilon_d^6$$

stands for Newton's constant of gravitation. With this, the constant radial displacement (5.9) of 'space' created by a spherically distributed total mass m can be written as

(7.4) $$\frac{a}{2} = \frac{Gm}{2c^2}$$

identifying 2a with the Schwarzschild radius of orthodox general relativity. A reformulation of (7.2)which includes relativistic motion of the test particle will be discussed in an Appendix to this paper.

It is one of the main results of ref. [1] that the parameters ε_d and r_o in (7.3) can be unambiguously expressed in terms of the fine-structure constant α and the classical electron radius r_e, namely by eqs. (1.4) to (1.7) of this paper. The result is

$$G = 6.6446 \times 10^{-11} m^3 kg^{-1} s^{-2} \tag{7.5}$$

to be compared with the accepted experimental figure of 6.6726×10^{-11}.

It remains to be proved that the geometrical considerations of Sections 4 and 5, concerning the match of a granular lattice ot continuum physics, do in fact warrant fractional-percent accuracy. However, in considering the 0.42% agreement, it should be borne in mind that the factors contributing to (7.3) span well over one hundred orders of magnitude between them. Thus, $r_o^2\,\varepsilon_d^6$ amounts to $5.3 \times 10^{-68}\, m^2$ while the coefficient c^3/h, which accommodates our empirical units of mass, length and time, equals 2.6×10^{59} m kg^{-1} s^{-2}. It should also be noted that there are no free parameters in a result that emerges as a necessary consequence of the unmodified model [1] which is supported by quantitative results of its own.

In the frame of this model, the Coulomb and gravitational forces are remote relatives in certain respects and quite different in others. As it turns out, both depend on the same pair of fundamental constants — α and r_e, known to us as ‘electric’ — and both can be traced to a volume perturbation of ‘space’ at the source. However, the two kinds of volume perturbation are fundamentally different and the difference excludes any mutual coupling. In the case of the electric charge this perturbation consists of (permanently) misplaced material, making it locally strong but bipolar, while the source of gravity exposed here is a dynamic density perturbation of oscillatory origin — exceedingly small but unipolar and therefore of sweepingly universal consequences. Also, the ‘Coulomb force’ described in ref. [1] depends on coupling to the boundary of the model universe while the ‘gravity’ of this paper is local.

Equations (7.1) to(7.3) fulfill the main aim of this paper, which is to present a mechanism for the source of gravity itself, as distinct from describing its effect in terms of curved space. The model does recover the essence of general relativity, the distortion of ‘space’ taking the form of strain in a Newtonian medium. Moreover, the generation of this strain by a source ‘mass’ is explained and related to known constants of nature rather than remaiing an *ad-hoc* assumption and an isolated experimental value. The action of the local acceleration of ‘space’(7.1) on the kinematics of waves therein goes beyond Newton’s law (7.2). It does, in fact, agree with general relativity — at least in a static spherical (Schwarzschild) configuration. This is sketched out in the Appendix.

APPENDIX
The Schwarzschild Metric

Seen from outside, this entire model is exactly governed by Netwonian dynamics and Galilean kinematics. Relativity is not an input but a result, namely the result of a transformation to the point of view of the model inhabitants. Special relativity has been treated in ref. [1]. General relativity, at least to the extent it is discussed here, turns out to have the same two components which are familiar from the orthodox description, namely a distortion of space and of time.

The first component emerges directly from one of the two main results of this paper, namely equation (7.4). It states that a spherically symmetric assembly of wave-centres with frequencies ω_i — called 'source mass' $m = \sum_i h\omega_i / c^2$ — creates a distortion of the surrounding wave-propagating medium (or ether). The distortion is characterized by a constant radial displacement (or shift of 'space')

$$s_r = \frac{a}{2} \tag{A-1}$$

where the constant

$$a = \frac{Gm}{c^2} \tag{A-2}$$

is evidently identical with half the Schwarzschild radius but G is now related to the rest of physics by (7.3) Thus, this distortion of 'space', as seen by the outside observer, creates a shift of all radii from r to $r_a = r + a/2$. Inhabitants, who insist that a surface of constant distance to the source equals 4π times their 'radius' squared, will use this new radius r_a as a scale length. In terms of the unperturbed outside observer's radius r the inhabitants' 'radius' $\bar{r}$ is, therefore, defined by

$$\bar{r} = \left(r + \frac{a}{2}\right)\frac{r_a}{r} = r\left(1 + \frac{a}{2r}\right)^2 \tag{A-3}$$

$$d\bar{r} = \left(1 - \frac{a^2}{4r^2}\right)dr \tag{A-4}$$

Secondly, the inhabitants' reference of time is also being changed. In an outsider's frame, moving with the test wave at its group velocity **v**, the ether surrounding the test-wave centre is observed to possess a velocity **u** = -**v** and a total acceleration

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial \mathbf{t}} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{Gm}{r^2}(1,0,0) \tag{A- 5}$$

the radial component being identical with $\ddot{s}$ of (7.1). As the situation is stationary in this frame, the partial derivative $\partial\mathbf{u}/\partial t$ is zero, leaving only the convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Therefore — and since only the radial component of (A- 5) is finite — the situation is described by

$$\frac{Gm}{r^2} = -\left(\frac{d\mathbf{u}}{dt}\right)_r = -\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right)_r = \left((\mathbf{v}\cdot\nabla)\mathbf{u}\right)_r = \frac{c^2}{\omega}\left((\mathbf{k}\cdot\nabla)\mathbf{u}\right)_r \tag{A-6}$$

where the wavenumber

(A-7) $$\mathbf{k} = \mathbf{v}\frac{\omega}{c^2} \quad ,$$

has been introduced in the last step. The scalar product $\mathbf{k}\cdot\nabla$ applied to the material velocity of a medium carrying a wave, produces a Doppler gradient of local frequencies (observed when at rest in the medium) so as to make the Doppler-shifted contributions at the wave-packet's centre all coincide with the common carrier frequency ω. Thus

(A-8) $$\left((\mathbf{k}\cdot\nabla)\mathbf{u}\right)_r = (\nabla\omega)_r = \frac{\partial\omega}{\partial r}$$

and therefore, because of (A-6) and (A-2)

(A-9) $$\frac{\partial\omega}{\omega\partial r} = \frac{a}{r^2}$$

(Note that, contrary to the main body of the paper, all frequencies ω mentioned here refer to test waves at radius r, the source being characterized by its total 'mass' m alone.)

Because of (A-7) and Hamilton's relation

(A-10) $$\frac{d\mathbf{k}}{dt} = -\nabla\omega \quad ,$$

universally valid in every dispersive wave,(A-9) merges into Newton's law of gravitation

(A-11) $$\frac{dv_r}{dt} = -\frac{c^2 a}{r^2} \quad ,$$

for $v_r << c$, hence $\omega \to \omega_o$. Note that the frequency gradient (A-8), (A-9) is a local property of the medium. It is *not* the change of carrier frequency (or 'gravitational redshift') — of opposite sign as shown by (A-10) — which is observed when the group motion is followed. The Doppler gradient (A-8) is the general mechanism for the interaction of a classical wave with its accelerated medium [4]. The mechanism does not act in addition to the seemingly obvious fact that for v << c the group acceleration equals the material acceleration of the medium; it is the explanation for that fact.

Integration of (A-9) yields a frequency potential ω(r) given by

(A-12) $$\frac{\omega_\infty}{\omega(r_a)} = e^{a/r_a}$$

without restriction to v << c. It is the essence of the explanation [1] of special relativity in this theory that the model inhabitants can have no other reference for time and distance but the spectrum of 'particle rest energies' $\hbar\omega_o$ *as they observe them* and of associated vacuum wavelengths c/ω_o. It follows that (A-12) has the universal and inescapable effect of shifting all the inhabitants' references of time by a factor $\exp(a/r_a)$ with respect to the model-constructors independent reference outside the model. This comes in addition to any special-relativistic Lorentz factor γ for constant relative motion. If the later transformations are tacitly included here, the transformation from the outside observer's time to the time of inhabitants in the gravitational field is given by (A-12) as

(A-13) $$t' = e^{a/r_a}\, t \quad .$$

At constant radius $r_a = r + a/2$ and hence at $\bar{r}$ as defined by (A-3), the inhabitants' time increment is, thus, given by

(A-14) $$d\bar{t} = e^{a/r_a}\, dt \quad .$$

Following their definition of ‘distance’ (the only one available to them) namely ‘time spent at nominal velocity c’, the inhabitants get for their infinitesimal tangential distance

(A-15) $$c d\bar{t} \equiv \bar{r} d\Omega = \left(1 + \frac{a}{2r}\right)^2 r\, d\Omega$$

where $d\Omega = \left(d\Theta^2 + \sin^2\Theta d\phi^2\right)^{\frac{1}{2}}$ is the polar angle seen from the initial position and (A-3) expresses $\bar{r}$ in terms of r.

For radial motion, on the other hand, the transformed time increment differs from $d\bar{t}$ and may be obtained as follows: let $t = r_a/v_r$ be the outside observer’s time for a signal (or indeed anything the inhabitants can recognize) to travel from the centre to the radius r_a at radial (group) velocity v_r. Then, because of (A-13) and $v_r = dr/dt$ the transformed time increment amounts to

(A-16) $$dt' = \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial r} dr = e^{a/r_a}\left(dt - \frac{a}{r_a^2}\frac{r_a}{v_r} dr\right) = \left(1 - \frac{a}{r_a}\right) d\bar{t} = \frac{1 - \frac{a}{2r}}{1 + \frac{a}{2r}} d\bar{t} \quad .$$

Insisting on spatial isotropy of time the inhabitants will apply the time scale $d\bar{t}$ from (A-14) — derived from the ‘rest-energy’ spectrum they observe at $\bar{r}$ as the only clock they can possibly employ — to their radial motion as well. Because of (A-16) their radial increment is, therefore, defined by

(A- 17) $$dr' \equiv c\, d\bar{t} = \frac{1 + \frac{a}{2r}}{1 - \frac{a}{2r}} c\, dt' \quad .$$

But $c dt'$ (not $c d\bar{t}$) is the actual radial increment $d\bar{r}$ covered at velocity c during the inhabitants’ time increment $d\bar{t}$. Substituting this into (A- 17) and expressing $d\bar{r}$ in terms of dr by (A-4) one finds

(A-18) $$dr' \equiv c d\bar{t} = \left(1 + \frac{a}{2r}\right)^2 dr \quad .$$

Equations (A-15), (A-16) and (A-18) squared give the inhabitants’ line element ds in four-dimensional space-time by

(A-19) $$ds^2 = -\left[\frac{1 - \frac{a}{2r}}{1 + \frac{a}{2r}}\right]^2 c^2 d\bar{t}^2 + \left(1 + \frac{a}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right)$$

in terms of the outside observer’s (or model constructor’s) unperturbed radius r (the inhabitants’ radius at which the test ‘particle’ would be found if the gravitation could be turned off). The co-ordinate time interval $d\bar{t}$ is what is shown by any local clock the inhabitants can employ or construct, whether it is at rest in the gravitational field or at infinite distance (*cf.* the discussion in ref. [5]).

Equation (A-19) is the isotropic form of the external Schwarzschild metric. Expressing r by the inhabitants radius $\bar{r}$ by means of (A-3) results in the standard form

(A- 20) $$ds^2 = -\left(1 - \frac{2a}{\bar{r}}\right)c^2 d\bar{t}^2 + \left(1 - \frac{2a}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega$$

in which the surface $\bar{r}$ = const. has the area $4\pi\bar{r}^2$.

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